Controlling Complex Contagions*

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May 14, 2026

Abstract

Many social and economic behaviors, from participating in a protest to adopting a new technology, require reinforcement from multiple peers to be worthwhile. The spread of these behaviors is often called *complex contagion*. We develop a tractable model of complex contagion on random networks, where agents take an action only when enough peers do too. We identify a tipping point in behavior: a level of network connectivity where participation (the fraction taking the action) jumps discontinuously. We characterize how the location and size of this tipping point depend on heterogeneity in connectivity among agents. When a principal with coarse control over the network structure likes connectivity but dislikes participation, she is pushed into "extreme" choices. She either chooses no participation, or a level significantly above the tipping point: to accept any participation is to accept a lot. The principal is "in for a penny, in for a pound".

JEL Codes: D85, Z13. *Keywords:* contagion, network design, random networks, threshold games, social norms.

*We thank Robert Akerlof, Nicholas Christakis, Krishna Dasaratha, Itay Fainmesser, Ben Golub, Kevin He, Richard Holden, and Matthew Jackson for helpful comments. We also thank participants at RINSE 2025, the Monash Network Economics Conference 2025, Network Science in Economics 2026, and Tilburg University. Financial support from the British Academy (SRG2425\251596) is gratefully acknowledged.

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1 Introduction

In many economic settings, an individual will take an action only when enough of their peers do as well. A citizen joins a protest only when a critical mass of friends will provide "safety in numbers"; a person adopts a workplace practice, new technology, or social norm only when enough peers have already done so. These behaviors spread through social networks via a process of mutual reinforcement, and are known as *complex contagions*.[1] They play a central role in settings ranging from political mobilization and corporate culture to technology adoption and the evolution of social norms.[2]

An important feature emerging from empirical work is that behavior in these settings can change suddenly in response to modest changes in the environment. Controlled network experiments show that a committed minority of almost 25% can flip a prevailing social convention. Just below this cutoff, change rarely occurs, while just above it, adoption of the new behavior spreads rapidly (Centola et al., 2018). "Tipping point" dynamics (Schelling, 1978)—where modest changes in early participation or the interaction environment produce outsized changes in outcomes—have also been observed in political mobilization (Madestam et al., 2013; Enikolopov et al., 2020).

These behaviors matter not only because they spread, but because they are often sustained inside interaction environments that some principal partly controls. Governments, firms, platforms, and regulators often value interaction among agents: it generates information, trade, collaboration, engagement, and learning. But the same interaction can also sustain coordinated behavior the principal dislikes. An authoritarian government benefits from citizens exchanging economically productive ideas, but may fear protest; a firm benefits from employees sharing best practices, but may be hurt by workplace norms that condone shirking; a platform benefits from user en-

[1]This language was introduced by Centola and Macy (2007) to describe contagion processes requiring reinforcement from multiple sources, in contrast to "simple" contagions that spread through a single contact. Our model captures this through an absolute participation threshold $k \geq 3$; see Section 7 for a discussion of alternative approaches.

[2]A substantial body of empirical work has documented the importance of social reinforcement in settings as varied as the adoption of new agricultural technologies (Bandiera and Rasul, 2006; Beaman et al., 2021), communication technologies (Björkegren, 2019), birth control use (Munshi and Myaux, 2006), health behaviors (Christakis and Fowler, 2007, 2008), protest participation (Larson et al., 2019; González, 2020; Bursztyn et al., 2021), and public health interventions (Airoldi and Christakis, 2024).

gagement, but may be harmed by coordinated harassment or manipulation. So a key consideration for the principal is how harmful coordination emerges as she changes the amount of interaction. If it emerges smoothly, the principal can respond smoothly at the margin. But if it emerges abruptly, she can be forced into all-or-nothing policies.

A limited empirical literature gives stark examples of principals restricting the channels through which coordinated behavior can spread. King et al. (2013) document that Chinese government censorship targets the coordination channels through which collective action can spread, preventing mass coordination against the regime. Internet shutdowns during protests and contested elections represent an even starker choice. They dramatically reduce connectivity: forgoing the economic benefits in order to suppress the *possibility* that coordinated collective action can arise (Access Now, 2023; V-Dem Institute, 2024). In response to viral misinformation, WhatsApp—a platform whose product is literally connection—capped message forwarding to five chats at a time (Melo et al., 2020). We do not read these examples as direct tests of our model. Rather, they motivate three ingredients that appear repeatedly in settings where principals worry about complex contagion: interaction is economically valuable, some collective behaviors require reinforcement from multiple peers, and principals often have only coarse control over the interaction environment.

This paper poses two questions. First, how does network structure shape coordinated participation in a complex-contagion environment? Second, how does a principal with coarse control optimally design the network?

Despite the accumulating evidence on tipping points and principals' responses to complex contagion, existing theories do not give us a useful map from network structure to behavior. Fixed-network analyses deliver rich microfoundations but struggle to deliver clean comparative statics with respect to network structure, while more tractable approaches push the network into the background. As a result, we still lack a simple benchmark that links explicit network structure to equilibrium behavior in complex contagion environments. This precludes clean analysis of when extreme policies are "forced" by the environment, and when more moderate policies are feasible.

We provide such a benchmark. Our model generates tipping points in agents' behavior directly from the interaction between local threshold-based preferences and global network structure. Crucially, we find that the network structure governs whether a principal is forced into extreme design choices. More concretely, we study a two-stage game between a principal, and a large population of agents who interact via a random network. Each agent has a heterogeneous propensity to interact, summarized by a CDF F which we call the *interaction profile*, and links form organically through chance interactions, not by design. Each agent then chooses whether to participate in a collective behavior, and prefers to participate if and only if at least $k \geq 3$ (the *participation threshold*) of his neighbors in the network do as well. The principal moves first: she controls a single lever, the *interaction rate* r, which determines the average number of links each agent has. But she cannot shape the microstructure of the realized network or the interaction profile. She values interaction directly, but bears a cost when agents participate. The principal therefore faces a fundamental tension: interaction is productive, but it also creates the capacity for collective action.

Our first set of results characterizes the relationship between network connectivity and equilibrium participation—the fraction of agents who participate in the collective action. We show that, under minimal regularity conditions on the distribution of propensities to form links and for any participation threshold $k \geq 3$, there is a critical cutoff for the level of connectivity: below this cutoff, participation is zero; above it, a positive fraction of agents participate. Participation emerges *discontinuously*—it jumps from zero to a strictly positive level. As connectivity increases beyond that critical cutoff, participation grows monotonically. Separately, the critical cutoff is increasing, and the level of participation decreasing in the participation threshold k. The more friends needed to "support" participation, the fewer agents will participate. To our knowledge, these comparative statics are novel. [3]

A key finding is that the *size* of the discontinuous jump depends on heterogeneity of agents' propensities to form links in an economically interpretable way. In a homogeneous benchmark where all agents form links at the same rate, the jump in participation when $k = 3$ is large: whenever a self-sustaining group exists, it must contain at least a quarter of all agents. We show that this large jump is robust to small amounts of heterogeneity in agents' propensities to form links: sufficiently homogeneous environments always produce large jumps. In contrast, in environments

[3]Formally, these results are derived by analyzing the giant k-core of the random network — the largest subgraph in which every agent has at least k connections to other members. The k-core is known to characterize equilibrium behavior in threshold games on finite, deterministic networks (Gagnon and Goyal, 2017; Langtry et al., 2024); we work with its large-network analogue. Although some of these comparative statics exist in the mathematics literature for the homogeneous (Erdös-Rényi) benchmark, to our knowledge they have not been established for the broader class of mixed-Poisson random graphs we study.

with strong heterogeneity—where a vanishingly small "core" of agents form links at a much higher rate than the rest—the jump can be made arbitrarily small. We provide an explicit example in which heterogeneity confines participation to a small "core".

Figure 1 provides a graphical illustration of these results. The left panel plots equilibrium participation as a function of network connectivity when the participation threshold is $k = 3$ for two environments: a homogeneous benchmark where all agents have the same propensity to form links (blue), and a highly heterogeneous core-periphery environment where one fifth of agents have a propensity of 5 and the remaining four fifths do not form links at all (red). The right panel shows the effect of varying the participation threshold k in the homogeneous benchmark.

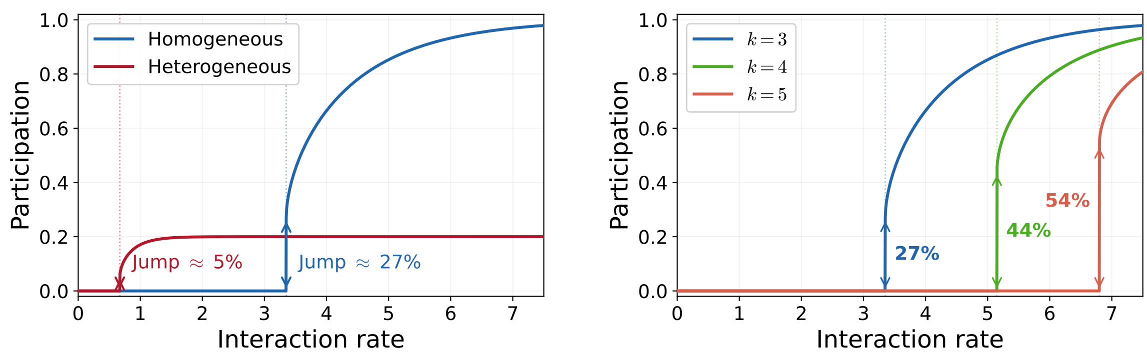

Figure 1: *Left:* Equilibrium participation as a function of network connectivity in two environments with the same mean degree: a homogeneous environment (blue) and a core–periphery environment (red). The arrows indicate the size of the jump—the minimum positive participation level. In the homogeneous environment, participation jumps by roughly 27%; in the core–periphery environment, by roughly 5%. *Right:* Equilibrium participation in the homogeneous benchmark for participation thresholds $k = 3, 4, 5$. Higher thresholds require more connectivity and produce larger jumps.

This clean characterization of equilibrium participation then opens up our second set of results. We characterize the principal's optimal choice of network connectivity and show how this depends on the heterogeneity of agents' propensities to form links. The presence—and more importantly, size—of the discontinuous jump in participation above the critical cutoff for network connectivity proves essential here. When the jump is large—as in homogeneous environments—the principal faces a significant "missing middle" in her strategy. To accept any participation is to accept a large amount of it: the discontinuous jump means that the principal either sets connectivity precisely at the critical cutoff (ensuring zero participation) or pushes well above it, where the benefits from connectivity are large enough to outweigh the costs of widespread participation. Intermediate levels of connectivity are never optimal. This is the stark, all-or-nothing logic that rationalizes extreme policy responses such as internet shutdowns.

But this extreme logic is not universal. When agents have highly heterogeneous propensities to form links, the jump above the cutoff can shrink, and with it, the missing middle. In a core–periphery environment, for instance, the missing middle vanishes as the "core" of highly connected agents becomes more concentrated. This is a key message of our paper: whether a principal facing a complex contagion is forced into extreme all-or-nothing policies depends on the heterogeneity of agents' link formation behavior. Homogeneous environments produce sharp tipping points and strong pressure toward extreme policies; heterogeneity can weaken that pressure by shrinking the jump above the cutoff.

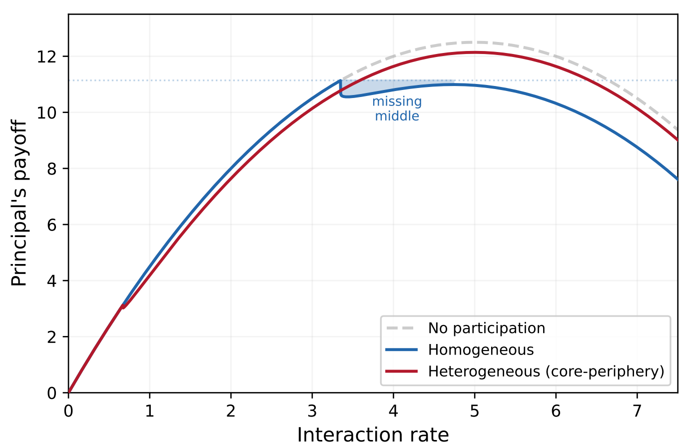

Figure 2: The principal's payoff in the homogeneous environment (blue) and core–periphery environment (red). The gray curve is the no-participation benchmark.

Figure 2 plots the principal's payoffs in the same two environments, and hence shows our second set of results graphically. The blue and red lines correspond to the homogeneous and core–periphery settings in Figure 1, respectively. The dashed gray curve shows the no-participation benchmark. When agents are homogeneous (blue line), there is a large drop in the principal's payoff at the critical cutoff. This is precisely because the jump is large when interactions are homogeneous. In turn, this leads to a large "missing middle", as the principal must choose a significantly higher level of connectivity—entailing a significantly higher level of participation—to recoup her losses from allowing any participation at all.

The rest of the paper is organized as follows. Section 2 discusses related literature. Section 3 sets out the model. Section 4 characterizes agents' equilibrium behavior and the discontinuous emergence of participation. Section 6 solves the principal's control problem, derives the missing-middle result, and links its severity to heterogeneity. Section 7 discusses directions for future work and concludes.

2 Related Literature

Our paper builds on and contributes to four main literatures.

First, we develop a tractable model of complex contagions on a random network, and derive sharp comparative statics for equilibrium behavior, which have been elusive in fixed-network models. There is a large literature that studies models of complex contagion, with applications to a very wide range of economic behaviors.[4] Existing work here primarily uses fixed networks (see, e.g. Gagnon and Goyal, 2017; Reich, 2023; Chandrasekhar et al., 2025), and often focuses on the diffusion of the action from some starting group of agents (Morris, 2000; Acemoglu et al., 2011).[5] More broadly, agents in our model play a game of strategic complementarities (Bulow et al., 1985; Milgrom and Roberts, 1990; Vives, 1990).

Second, by developing a model of collective behavior on a random network, we connect to a growing literature on simple contagion—where agents need only *one* neighbor to act—on random networks (Campbell, 2013; Akbarpour et al., 2020; Campbell et al., 2024a,b; Langtry, 2025). Sadler (2020) studies a general class of diffusion games on configuration model random graphs that are closely related to ours. A key distinction is that simple contagions produce continuous emergence of viral behavior: the giant component grows smoothly as connectivity increases (Centola and Macy, 2007; Centola, 2010). Complex contagions, by contrast, produce *discontinuous* transitions—participation jumps from zero to a positive fraction at the critical cutoff.

[4] These include technology adoption (Reich, 2023), protest (Chwe, 2000), pricing (Zhang, 2025), financial contagion (Rogers and Veraart, 2013; Elliott et al., 2014), persuasion (Candogan, 2022), and choices of whether to participate in formal markets (Gagnon and Goyal, 2017). Granovetter (1978) and Schelling (1978) also suggest many other applications.

[5] Another approach has been to push the network into the background, by abstracting from it altogether (Granovetter, 1978; Schelling, 1978) or imposing a "mean field" assumption (Jackson and Yariv, 2006; López-Pintado, 2006, 2008), or to assume that agents only have partial information (Galeotti et al., 2010; Leister et al., 2022).

Our model makes this contrast precise and tractable on the same class of random networks. Campbell et al. (2024b) also study a mixed Poisson random graph in a model of entry deterrence and pricing, focusing on the role of the giant component.

Third, we contribute to a literature on discontinuous phase transitions in networks—settings where small changes in network structure induce large, discontinuous changes in outcomes. Watts (2002) provides the seminal analysis of fractional-threshold contagion on random networks, showing that cascades can fail entirely or succeed on a large scale, with little in between. Elliott et al. (2022) show how a need for multiple types of input in supply networks can produce analogous discontinuous transitions, and provide comparative statics for the fragility of such networks. In our model, the discontinuity arises directly from the nature of threshold-based complex contagion itself: agents require a fixed number of participating neighbors, and this creates a sharp tipping point in aggregate behavior. We provide a number of new comparative statics and characterize how the distribution over agents' propensities to interact shapes the size and location of the discontinuity—including a comparison principle identifying when greater heterogeneity lowers the critical cutoff. Jackson and Storms (2026) also study complex contagion on random networks using stochastic block models, but focus on identifying which groups must take the same action and on optimal seeding, rather than on comparative statics for aggregate participation.[6]

Finally, we embed our analysis within a principal's optimal network design problem where the principal chooses aggregate connectivity. We show that homogeneity-versus-heterogeneity in the network structure influences whether the optimal policy is always extreme or can be interior. This connects us to the literature on network design and formation. This literature has increasingly focused on settings where a network forms and then agents play a game on the resulting network (Galeotti and Goyal, 2010; Sadler and Golub, 2021; Kinateder and Merlino, 2017, 2022). In part due to technical challenges, there is little work on network formation in a random-network setting. Elliott et al. (2022) and Langtry (2025) are exceptions, but consider endogenous formation by agents rather than control of overall connectivity by a principal. Galeotti et al. (2020) study the design of interventions in networks where a planner targets agents based on their degree. In a significantly different setting, Acemoglu

[6] Their goal is to understand (a) which groups of agents must take the same action in any equilibrium, and (b) how to use this knowledge to best pick a set of initial adopters to maximize the spread of a new behavior. Our focus is instead on comparative statics for overall equilibrium behavior with respect to features of the network, as well as how a principal chooses overall network connectivity.

et al. (2024) study platform design in a misinformation context and also find that the optimal policy can take a bang-bang form: the platform either allows wide sharing or restricts it sharply. Our analysis complements theirs by showing that this extreme logic is not universal—it depends on the heterogeneity in agents' propensities to form links.

3 Model

We consider a sequence of games $\{\Gamma^{(n)}\}_{n\in\mathbb{N}}$ indexed by the number of agents n. We now describe the structure of a specific game $\Gamma^{(n)}$.

Agents, Actions & Timing. There is a single principal ("she"), P, and n agents ("he"), indexed $i \in N \equiv \{1, \ldots, n\}$. In the first period ($t = 1$), the principal chooses an *interaction rate* $r \geq 0$, taking an *interaction profile* F—a distribution over agents' propensities to interact, defined formally below—as given.[7] Then Nature forms a large random network among agents (which we describe shortly). In the second period ($t = 2$), each agent simultaneously chooses whether or not to take a binary action, $a_i \in \{0, 1\}$. For clarity, we will say that an agent who takes action 1 *participates*, and an agent who takes action 0 *abstains*.

Random network formation and interaction profile. The principal controls only the interaction rate r (i.e. the network connectivity). But agents differ in their propensity to interact and form links. Each agent i draws an i.i.d. non-negative interaction weight W_i from a distribution F. This captures heterogeneity in interactions: i's expected degree is $r \times W_i$.[8] The interaction weights W_i capture this variation. We impose the following condition on the interaction profile.

Assumption 1. *Weights* $(W_i)_{i=1}^n$ *are i.i.d. with* $W_i \sim F$*, satisfying* $\mathbb{E}[W] = 1$*, and* $\mathbb{E}[W^2] < \infty$.[9]

[7] We view F as a slow-moving, difficult-to-change feature of the environment, while r can be manipulated. Exploring what happens when the principal can also shape F is an interesting direction for future work; see Section 7.

[8] Such heterogeneity is well-documented empirically: degree distributions in real-world networks typically have a small fraction of agents maintaining many more connections than the median (Jackson, 2008).

[9] While mild, this can be violated by sufficiently heavy-tailed profiles. We discuss this in Online Appendix OA.3.

The assumption that $\mathbb{E}[W] = 1$ is simply a normalization. Conditional on W_i and r, each agent draws a number of stubs (their degree) $D_i \mid W_i \sim \text{Poisson}(rW_i)$ independently across i. Then conditional on the realized degree sequence $(D_i)_{i=1}^n$, Nature forms a (simple) random graph via the *configuration model*: stubs are paired uniformly at random, and we condition on the resulting graph being simple (i.e. no self-loops or multiple edges).[10] We let $G^{(n)}$ denote the adjacency matrix of a realization of this random graph: $G_{ij}^{(n)} \in \{0, 1\}$.

Preferences: principal. The principal values interactions between agents, but bears a cost when agents *participate* in the undesirable behavior. Providing the underlying interaction capacity—opportunities to engage, communication infrastructure, meeting spaces—is also costly. We capture the principal's net benefit from interactions, before accounting for participation, with a simple linear-quadratic form: $\alpha r - \frac{1}{2}r^2$, where $\alpha > 0$ is the marginal benefit of interaction and $\frac{1}{2}r^2$ a convex cost. Online Appendix OA.2 microfounds the linear benefits in r as the equilibrium value of a game where agents take bilateral actions with each of their friends, and the principal benefits from these actions.[11] Let $\beta > 0$ denote the per-capita cost she bears when agents participate, and $\bar{a} = \frac{1}{n}\sum_i a_i$ denote the *participation rate*. The principal's per-capita payoff at interaction rate r is

$$\pi(r, F, \bar{a}) \;=\; \alpha\, r \;-\; \beta\, \bar{a} \;-\; \frac{1}{2}r^2. \tag{1}$$

Note that the interaction profile F only affects the principal's objective through equilibrium participation.

The two cost terms have distinct economic interpretations. *Infrastructure cost* ($\frac{1}{2}r^2$): the cost of providing interaction opportunities at rate r, regardless of how connections are distributed. This is an *extensive-margin* cost: it depends on the total volume of interaction, not how those interactions are distributed across agents. *Participation cost* ($\beta\bar{a}$): the cost the principal bears when agents participate.

[10]Under Assumption 1, the configuration model produces a simple graph with asymptotically positive probability, so this conditioning is without loss.

[11]But neither the linearity of the benefit nor the quadratic cost are essential: our qualitative results only require that the principal's benefit, absent participation costs, is twice continuously differentiable and strictly concave with an interior maximum.

Preferences: agents. Agents' actions are strategic complements: participating is more attractive to agent i when more of his neighbors participate. Specifically, i prefers to participate (choose $a_i = 1$) if and only if at least $k \geq 3$ of his neighbors also participate, where k is the *participation threshold.* Agents' preferences can be represented by a utility function $u_i(a_i, M_i)$ such that:

$$u_i(1, M_i) \geq u_i(0, M_i) \quad \Longleftrightarrow \quad M_i \geq k, \tag{2}$$

where $M_i = \sum_{j \neq i} G_{ij}^{(n)} a_j$ is the number of participating neighbors of i.

Information. For expositional simplicity, we assume that the realization of the network is common knowledge to agents. This means that actions a_i are functions $a_i = a_i(G^{(n)})$ of the realized network $G^{(n)}$ observed by agents. In Online Appendix OA.1 we provide a microfoundation which demonstrates that our model here can be viewed as the reduced-form of a model in which agents participate in an explicit diffusion process and only observe the actions of their neighbors.

Equilibrium. A strategy profile $\left(r^*, a_1^*, a_2^*, \ldots, a_n^*\right)$ is a *subgame perfect Nash equilibrium* of the game $\Gamma^{(n)}$ if, for any realization of the graph $G^{(n)}$, we have:

$$\text{for all } i \in N, \quad a_i^* = 1 \iff u_i(1, M_i^*) \geq u_i(0, M_i^*),$$

and the principal chooses r^* such that

$$r^* \in \arg\max_{r \geq 0} \; \mathbb{E}_{\mathbb{P}_{r,F}}\big[\pi(r, F, \bar{a}^*)\big],$$

where $\bar{a}^* = \frac{1}{n} \sum_{i=1}^{n} a_i^*$ is the equilibrium level of participation in the second stage and $\mathbb{P}_{r,F}$ is the distribution over networks generated by the principal's choice of r and the interaction profile F.

This definition requires that (i) agents best-respond to other agents given the realized network, and (ii) the principal maximizes her expected payoff, anticipating both the network structure and the resulting actions. In this paper we focus on the largest (i.e. highest participation) equilibrium, because it captures the worst feasible self-sustaining outcome for the principal and is also selected by natural local best-response dynamics. Online Appendix OA.1 formalizes the latter interpretation.[12]

4 How Agents Behave

We begin with the second stage of the game, and characterize how agents' behavior depends on the interaction rate r and the interaction profile F. The second stage is a threshold game played on the realized network G. So before going further, it is important to discuss the role of the *k-core*—a notion of "cohesive" parts of a network—in characterizing equilibrium behavior.

4.1 Equilibrium and the k-core

In threshold games on networks, the largest equilibrium is determined by the network's "core structure". The k-core of a graph G is the largest induced subgraph H of G such that every agent in H has degree at least k within H (Seidman, 1983). In our model the set of agents who participate in the largest equilibrium is precisely the k-core of the network G.

Remark 1. *In the largest equilibrium, agent i participates ($a_i = 1$) if and only if he belongs to the k-core of the realized network $G^{(n)}$.*[13]

We can see why this is the case by considering how agents might reason about a stable outcome. Any agent with fewer than k neighbors in the entire network knows he can never meet the participation threshold. So he will choose to abstain. Knowing this, other agents can revise what they expect their neighbors to do. An agent who initially had k neighbors might now expect fewer than k to participate, causing him to also abstain. This iterated removal of agents who lack sufficient support continues until only a stable group remains. The agents left are exactly those in the k-core; each has at least k connections to others who are also participating.

[12] Agents' actions are strategic complements, so the second stage of the game on any realized network $G^{(n)}$ has a *complete lattice* of Nash equilibria under the component-wise order on $\{0,1\}^n$ (Milgrom and Roberts, 1990; Topkis, 1998). In particular, there exists a *largest* equilibrium $a^{\max}(G^{(n)})$—the one with the most participating agents.

[13] The direct link between the largest equilibrium and the k-core has been established in prior studies of threshold games on fixed (finite) networks (Gagnon and Goyal, 2017; Langtry et al., 2024).

A key implication of Remark 1 is that characterizing equilibrium behavior in our model amounts to characterizing the fraction of agents belonging to the k-core of the random network induced by (r, F). The challenge in fixed-network models has been that it is very difficult to make strong predictions about how equilibrium behavior changes as a function of the primitives of the network. Using random networks allows us to obtain clean results in the limit of a large population. Therefore the remainder of our analysis focuses on the asymptotic properties of the model as the number of agents $n \to \infty$. As such, the equilibria we characterize are limits of subgame perfect equilibria of the finite games $\Gamma^{(n)}$.[14] We will often omit the limit in our statement of results, but any such results should be understood as applying with high probability as $n \to \infty$.

4.2 Equilibrium behavior

Our characterization of equilibrium behavior in the second stage builds on established results from random graph theory. The existence of a *giant k-core* (a k-core containing a constant fraction of agents with high probability as $n \to \infty$) is known to depend on a sharp cutoff condition.[15] Above this cutoff, the size of the giant k-core is strictly increasing in network connectivity. This provides the crucial link between the interaction rate r and equilibrium participation.

To fix ideas, let $a_k(r; F) \equiv \lim_{n\to\infty} \frac{1}{n} \sum_i a_i^*$ denote the limiting fraction of participating agents—the (level of) participation—given an interaction rate r and profile F. It is convenient to define $\psi_m(x) \equiv \mathbb{P}(\text{Poisson}(x) \geq m)$, and the functions

$$\Phi_k(x) \equiv \mathbb{E}[W\, \psi_{k-1}(Wx)]\,, \qquad A_k(x) \equiv \mathbb{E}[\psi_k(Wx)]\,. \tag{3}$$

When $W \equiv 1$ (the Erdős–Rényi benchmark), these reduce to $\Phi_k(x) = \psi_{k-1}(x)$ and

[14] All references to "equilibrium" from here on should be taken to mean "limits of equilibria in the sequence of finite games".

[15] For Erdős–Rényi graphs—which, in our model, corresponds to the homogeneous-interaction case—the canonical reference is Pittel et al. (1996); for configuration models with general degree distributions, see Janson and Luczak (2007); Janson and Łuczak (2008). The critical cutoff condition is usually referred to as the critical *threshold* in this mathematics literature. But to avoid confusion, we reserve the term "threshold" for the participation threshold k.

$A_k(x) = \psi_k(x)$. Then define the critical cutoff

$$c_k(F) \equiv \inf_{x>0} \frac{x}{\Phi_k(x)}. \tag{4}$$

Under Assumption 1 and $k \geq 3$, $\Phi_k(x) = o(x)$ as $x \downarrow 0$ and $\Phi_k(x) \leq \mathbb{E}[W] = 1$ for all x, so $x/\Phi_k(x) \to \infty$ as $x \downarrow 0$ and as $x \to \infty$; hence the infimum in (4) is attained at some finite $x_k^\star(F) > 0$.[16] For expositional simplicity, we assume that the minimizer of $x/\Phi_k(x)$ is unique and satisfies a mild regularity condition.

Assumption 2. *$g(x) \equiv x/\Phi_k(x)$ has a unique minimizer and $g'(x) > 0$ for all $x > x_k^\star(F)$.*

This assumption ensures that the k-core emerges through a single discontinuous jump.[17] Our assumption always holds when the interaction profile is sufficiently homogeneous (such as the Erdős–Rényi benchmark; see Online Appendix OA.8.5 for the formal statement), and generally holds when heterogeneity is sufficiently "nice". In Online Appendix OA.8.2 we provide a single-crossing condition on F that is sufficient for Assumption 2 to hold. We also discuss the more general analogues to our main results. Unless explicitly stated otherwise, all results throughout the rest of the paper operate under Assumptions 1 and 2.

Next, let $x_k(r)$ be the largest solution to

$$x = r\,\Phi_k(x) = r\,\mathbb{E}[W\,\psi_{k-1}(Wx)]\,. \tag{5}$$

Our first result shows that whether equilibrium participation is positive turns on whether $r \lessgtr c_k(F)$. The exact behavior of the finite graph at $r = c_k(F)$ depends on how it behaves in the "critical window" and is not important for our analysis. We

[16]For completeness, Lemma 2 in Online Appendix OA.8.1 shows $\Phi_k(x) = o(x)$ as $x \downarrow 0$.

[17]Without Assumption 2, g can have multiple local minima, in which case the giant k-core emerges through several disjoint jumps as connectivity increases. We discuss this case in Online Appendix OA.8.2.

therefore define limiting participation at the cutoff by its left limit:[18]

$$a_k\left(c_k(F);F\right) \equiv \lim_{r\uparrow c_k(F)} a_k(r;F) = 0.$$

With this in place, we are now ready to characterize equilibrium behavior.

Theorem 1 (Participation). *Fix $k \geq 3$ and an interaction profile F. Equilibrium participation:*

(i) is positive if and only if the interaction rate is sufficiently high ($r > c_k(F)$);

(ii) for every $r > c_k(F)$, is given by

$$a_k(r;F) = A_k(x_k(r)) = \mathbb{E}[\psi_k(W\,x_k(r))]\,, \tag{6}$$

where $x_k(r)$ is as in (5).[19]

Theorem 1 establishes a sharp tipping point in equilibrium behavior. Unless the interaction rate r exceeds the critical cutoff $c_k(F)$, the network is too sparse to support a giant k-core, and equilibrium participation is zero. Above this cutoff, a strictly positive fraction of agents participate, characterized by part (ii). As we show in Section 5, this transition is not gradual—participation *jumps* discontinuously from zero to a strictly positive level at the cutoff.

The intuition for this lies in the nature of social reinforcement required for participation. For any group of agents to form a stable, self-sustaining equilibrium (i.e., to be a k-core), each member must have their participation supported by at least k friends who are also part of that same group.

This stands in sharp contrast to the *continuous* emergence of the giant component—a related feature of large random networks that has been shown to play a key role in

[18]This convention is without loss for the principal's limiting design problem. For every $\varepsilon > 0$, the interaction rate $r = c_k(F) - \varepsilon$ yields zero limiting participation, while the direct interaction payoff converges to its boundary value as $\varepsilon \downarrow 0$. Hence, in any environment where $r = c_k(F)$ generates a giant k-core with positive asymptotic probability, a principal with $\beta > 0$ would avoid this by choosing just below the cutoff at arbitrarily small direct-payoff cost. We therefore make no probabilistic claim about the finite graph at $r = c_k(F)$; all references below to choosing the cutoff should be read as choosing this "no participation boundary".

[19]Exactly at the cutoff $r = c_k(F)$, the fixed-point equation admits a tangency solution, but the k-core law of large numbers (Proposition 4.1 of Janson (2009)) does not apply there. Accordingly, we define $a_k(c_k(F);F) = 0$ by convention and study the right-limit $J_k(F) = \lim_{r\downarrow c_k(F)} a_k(r;F)$.

driving behavior in models where the network spreads information (see, for example, Dasaratha, 2023; Campbell et al., 2024b). Unlike the giant k-core, the giant component is well studied in economics (see Jackson, 2008; Easley et al., 2010; Goyal, 2023, for textbook treatments).[20]

4.3 Comparative statics: how agents respond to the environment

We now consider how equilibrium participation varies with the interaction rate r and the participation threshold k. The level of participation is increasing in the interaction rate and decreasing in the participation threshold.

Theorem 2 (Comparative statics: monotonicity)**.** *Fix an interaction profile F.*

(i) (Monotonicity in r.) *For every fixed $k \geq 3$, equilibrium participation, $a_k(r; F)$, is constant in r when $r < c_k$, and strictly increasing when $r > c_k$.*

(ii) (Monotonicity in k.) *For every fixed $r \geq 0$, equilibrium participation, $a_k(r; F)$, is constant in k when $r < c_k$, and strictly decreasing when $r > c_k$.*

Intuitively, as the interaction rate increases, the degree distribution shifts to the right—agents have more neighbors on average. This increases the likelihood that the number of an agent's neighbors who participate will exceed k, reinforcing participation. Conversely, a higher threshold k means fewer agents will have sufficient participating neighbors, causing them to abstain. This initial abstention can then induce other agents to do the same, ultimately reducing participation.[21] Formally, these results are unlocked by our characterization in Theorem 1. To our knowledge, these comparative statics were not previously known.

A particular point of interest is the jump in participation due to the discontinuous emergence of the giant k-core. We now examine this jump in more detail. It is both of intrinsic interest—little is currently known about the jump beyond its existence—and will prove important for understanding the principal's optimal behavior.

[20]The giant component is closely related to the $k = 2$-core. Taking the giant component and removing all agents with degree 1 yields the 2-core. In the mixed-Poisson model that we use, the critical cutoff for the emergence of both the 2-core and the giant component is $r = 1/\mathbb{E}[W^2]$, which collapses to $r = 1$ in the Erdös-Rényi case.

[21]In contrast, when there is zero participation in equilibrium, changes in the network connectivity and/or the participation threshold have no impact. This follows immediately from Theorem 1(i).

5 The Jump: Location and Size

We now examine the location of the jump—the critical cutoff $c_k(F)$—and the *size* of the jump at that cutoff. Together these determine when participation first becomes possible and how much participation must arise when it does. We show that both are economically important, and characterize how they depend on the interaction profile F and on the participation threshold, k. But before stating any results, we need to define the size of the jump formally. It is the right-limit of participation at the critical cutoff:

$$J_k(F) \equiv \lim_{r \downarrow c_k(F)} a_k(r; F). \tag{7}$$

With this in place, we can now confirm that the jump is in fact a jump. That is, that above the critical cutoff, the participation rate is bounded away from zero—no matter how close we get to the critical cutoff.

Proposition 1. *Fix any $k \geq 3$ and any interaction profile F.*

$$J_k(F) > 0.$$

As the interaction rate increases continuously, there is always a discontinuous jump from zero participation to some positive amount. Intuitively, this is because agents need *several* friends to take the action alongside them. When the interaction rate is low (below the critical cutoff), some set of agents may have at least k friends. And some subset of them may have at least k friends who in turn have at least k friends. But not enough of those friends will themselves have at least k friends who have k friends, and so on. There will be some "weak links" preventing mutually sustaining actions. So nobody participates. As the interaction rate rises just past the critical cutoff, those "weak links" will gain enough friends. This will be the "final piece of the puzzle" for a significant group of agents—leading to this whole group now being able to sustain participation. Hence there is a jump.

But is this group of agents large relative to the overall population? It turns out that this depends critically on the heterogeneity of the interaction profile, F.

5.1 The role of heterogeneity

When interactions are homogeneous (i.e. when $W \equiv 1$), the jump involves at least one quarter of the whole population. So the jump is *economically* significant. Further, this is *not* a knife-edge case reliant on there being no heterogeneity. The jump remains large when the interaction profile F has some heterogeneity.

Theorem 3. *Fix any $k \geq 3$. Let δ_1 denote the degenerate interaction profile with $W \equiv 1$.*
(i) (Homogeneous case.) *If $W \equiv 1$, then*

$$c_3 \approx 3.35, \quad \text{and} \quad J_3(\delta_1) \approx 0.268.$$

(ii) (Limited heterogeneity.) *If the variance of the interaction profile F is small enough, then both the critical cutoff and the size of the jump are close to those of the homogeneous case. Formally, for every $\eta > 0$ there exists $\varepsilon > 0$ such that for any F with $\sqrt{\mathrm{Var}(W)} \leq \varepsilon$, we have:*

$$|c_k(F) - c_k(\delta_1)| < \eta \quad \text{and} \quad |J_k(F) - J_k(\delta_1)| < \eta.$$

Notice that Theorem 3(i) formally characterizes the case where $k = 3$. But it is easy to verify that the size of the jump in the homogeneous case gets bigger as the participation threshold k rises. For example, when $k = 10$, $J_{10}(\delta_1) \approx 0.74$, and when $k = 100$ it is around 0.95 (see Figure 4).

Recent empirical work has identified a critical mass of approximately 25% as a "tipping point" required for a committed minority to overturn an established social norm (Centola et al., 2018). This highlights the necessary conditions for *initiating* a large-scale behavioral cascade. Theorem 3 provides a complementary theoretical perspective by characterizing the conditions for the resulting behavior to be *self-sustaining*: under a moderate participation threshold ($k = 3$), at least a quarter of the population must participate in the homogeneous benchmark. Our result suggests a possible explanation for this empirical tipping point: individuals may be willing to adopt a new norm only when they perceive the movement has enough momentum to become self-sustaining.

However, this large jump is not universal. It relies on interactions between agents being somewhat homogeneous. Sufficient heterogeneity can make the jump arbitrarily small.

Remark 2. *For every $\varepsilon > 0$ there exists a profile F with $J_k(F) < \varepsilon$.*

We now show how one can construct such a profile F by concentrating all of the interactions in an increasingly smaller group of agents. For any interaction profile F with weights W, we define a *diluted* interaction profile, $F^{(\nu)}$ for any $\nu \in (0, 1]$ with weights

$$W^{(\nu)} \equiv \frac{B}{\nu} W,$$

where $W \sim F$ and $B \sim \text{Bernoulli}(\nu)$ are independent. This dilution is a mean-preserving spread of F.[22] In expectation, a fraction $1 - \nu$ of agents will not interact at all, and the remaining fraction ν will have their interaction rate scaled up by a factor $1/\nu$. In this special case, everything scales linearly.

Remark 3. *If F satisfies Assumption 2, then so does $F^{(\nu)}$, and for all $k \geq 3$*

$$c_k(F^{(\nu)}) = \nu\, c_k(F) \qquad \textit{and} \qquad J_k(F^{(\nu)}) = \nu\, J_k(F).$$

This shows how heterogeneity can shrink the size of the jump not by changing the within-core reinforcement technology, but by restricting the set of agents who can plausibly form a mutually reinforcing group. The network may be extremely dense *within* a small active set (mean degree r/ν), while the overall participation fraction scales with the size ν of that set. Moreover, a more heterogeneous environment has a lower cutoff: $c_k(F^{(\nu)}) = \nu\, c_k(F)$. As the active fraction shrinks, participation becomes possible at lower interaction rates—the network needs less overall connectivity to trigger collective action among the remaining active agents.[23]

This creates a tension for the principal. This more heterogeneous environment lowers the cutoff (participation is possible sooner) but also shrinks the jump (the discrete cost from crossing the cutoff is smaller). Whether this tension resolves in the principal's favor depends on parameters; Section 6 characterizes the tradeoff precisely through the missing-middle gap $d_k(F)$, which depends on both $c_k(F)$ and $J_k(F)$.

Numerical methods show the same tension is present when the interaction profile follows a gamma or lognormal distribution—parametric families commonly used in

[22]Formally, $\mathbb{E}[W^{(\nu)} \mid W] = \mathbb{E}[B/\nu] \cdot W = W$, so by Jensen's inequality $\varphi(W) \leq \mathbb{E}[\varphi(W^{(\nu)}) \mid W]$ for any convex φ; taking expectations gives $\mathbb{E}[\varphi(W)] \leq \mathbb{E}[\varphi(W^{(\nu)})]$, i.e., $F \leq_{\text{cx}} F^{(\nu)}$.

[23]In Online Appendix OA.5 we show that Remark 3 is robust to relaxing the assumption that peripheral agents have no links.

applied network analysis. Figure 3 plots the critical cutoff c_k and the size of the jump J_k as a function of the dispersion for both families. For these families, more heterogeneity both lowers the cutoff and shrinks the size of the jump. This is suggestive of a broader pattern: for regular parameterized families, greater heterogeneity tends to lower the cutoff *and* shrink the jump.

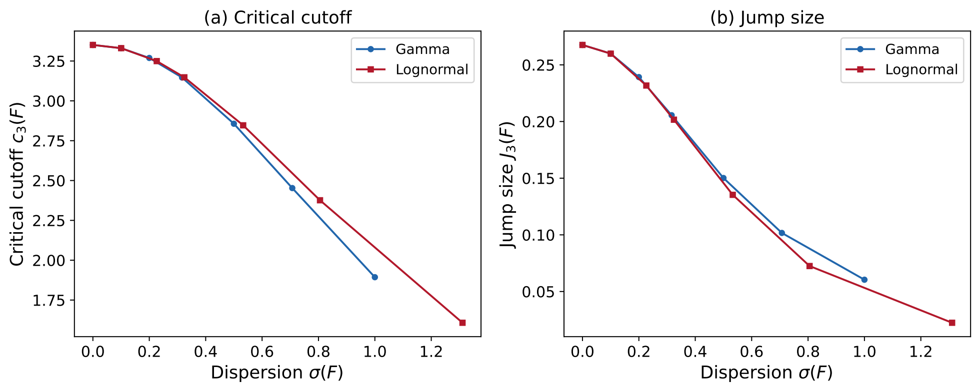

Figure 3: Critical cutoff $c_3(F)$ (left) and jump $J_3(F)$ (right) against dispersion $\sigma(F) = \sqrt{\mathrm{Var}(W)}$, for the gamma and lognormal families. Both families show the same pattern: increasing heterogeneity lowers the cutoff and shrinks the jump.

But is this *universally* true? Does more heterogeneity always lower the cutoff? It turns out that the answer is no. The convex order alone need not be enough to determine the effect of a change in the interaction profile. What matters is how a mean-preserving spread shifts mass around the margin of self-sustaining participation, which is determined by the threshold k. In light of this, we now provide a sufficient condition for a general change in the interaction profile F to reduce the critical cutoff c_k. It is not needed for characterizing the principal's behavior in Section 6, but is of independent interest as it provides a tool for understanding how behavior in threshold games responds to complex changes to the network structure.

5.2 A comparison principle for the critical cutoff

It is convenient to define, for $t \geq 0$,

$$C_F(t) \equiv \mathbb{E}[(W - t)_+]\,. \tag{8}$$

This measures the expected "excess interaction weight" that remains after stripping away the first t units of each agent's weight. In the finance literature, C_F is known as the *call-price curve* of the random variable W. For equal-mean profiles, ordering profiles by convex order is equivalent to ordering these curves pointwise: $F_1 \leq_{\text{cx}} F_2$ if and only if $C_{F_1}(t) \leq C_{F_2}(t)$ for all $t \geq 0$.

We can write Φ_k as a weighted integral against the call-price curve. So for every $x > 0$,

$$\Phi_{k,F}(x) = \frac{x^{k-1}}{(k-2)!} \int_0^\infty C_F(t)\, t^{k-2} e^{-xt}(k - xt)\, dt. \tag{9}$$

An analogous expression can be given for A_k. Together, Φ_k and A_k (from eq. (5) and eq. (6), respectively) pin down the size of the giant k-core. We derive both of these expressions formally in Online Appendix OA.7.

Importantly, the term $t^{k-2}e^{-xt}(k - xt)$ in eq. (9) changes sign exactly once—at $t = k/x$. This sign change is the key to understanding which forms of heterogeneity move the cutoff, and which do not. Recall that at the equilibrium $x = x_k(r)$, so an agent with weight w has wx participating neighbors in expectation. So an agent with weight $w = k/x$ expects exactly k participating neighbors—right at the participation threshold. Agents below this weight fall just short of the participation threshold; agents above it are safely inside the giant k-core.

With this, we can now ask: when does a mean-preserving spread in F lower the critical cutoff? For some initial interaction profile F_1, there is an associated value of x, denoted $x^\star_{1,k}$, that is the level of x at the critical cutoff $c_k(F_1)$. In turn this implies a threshold weight $w^\star_{1,k} := k/x^\star_{1,k}$ such that agents with higher weights are able to sustain participation when connectivity is at the cutoff, while those just below are marginal—close to, but below the participation threshold. Intuitively, agents with weights at $w^\star_{1,k}$ are those who just make it into the giant k-core at the point where it emerges. Since the kernel in (9) changes sign at the threshold weight and decays exponentially away from it, the balance of the call-price curve on either side of $w^\star_{1,k}$ determines which direction the cutoff moves. Theorem 4 makes this precise: whether a mean-preserving spread raises or lowers the critical cutoff depends on how it shifts weight in the call price curve around $w^\star_{1,k}$.

Theorem 4 (Critical cutoff comparison)**.** *Fix $k \geq 3$. Let $F_1 \leq_{\text{cx}} F_2$, so that $D(t) \equiv C_{F_2}(t) - C_{F_1}(t) \geq 0$ for all t. Suppose F_1 satisfies Assumption 2, and let $x^\star_1 \equiv x^\star_k(F_1)$*

denote the unique minimizer of $g_{F_1}(x) = x/\Phi_{k,F_1}(x)$. *If*

$$\int_0^\infty D(t)\, t^{k-2} e^{-x_1^\star t}(k - x_1^\star t)\, dt \geq 0, \tag{10}$$

then $c_k(F_2) \leq c_k(F_1)$. *In particular, a sufficient condition for* (10) *is* $D(t) = 0$ *for all* $t > k/x_1^\star$.

Extra dispersion among agents just below the margin of self-sustaining participation—those whose expected participating neighbors fall short of k—raises Φ_k and lowers the cutoff. This is because spreading mass among underconnected agents has increasing returns, creating some who become newly capable of sustaining participation. Formally, this is driven by the fact that $w \mapsto w\,\psi_{k-1}(xw)$ is convex in this region. In contrast, extra dispersion among agents already safely above the threshold has little effect: these agents participate regardless, so redistributing weight among them does not create new participants.

Theorem 4 clarifies why the critical cutoff cannot be ranked by the convex order alone. The integral condition (10) asks whether, evaluated at the minimizer $x_1^\star$ under F_1, the positive contribution to Φ_k from agents below the threshold weight outweighs the negative contribution from those above it. Our sharper sufficient condition—that $D(t) = 0$ for all $t > k/x_1^\star$—is the special case where the negative side vanishes entirely: all the extra dispersion under F_2 is concentrated among agents who fall short of the participation threshold. Note that neither condition requires solving for the equilibrium under F_2.[24]

Theorem 4 also makes it immediately clear why Bernoulli dilution behaves so cleanly. The identity $C_{F^{(\nu)}}(t) = C_F(\nu t)$ yields $\Phi_{k,F^{(\nu)}}(x) = \Phi_{k,F}(x/\nu)$ and hence $c_k(F^{(\nu)}) = \nu\, c_k(F)$ immediately. Bernoulli dilution is special because everything rescales together—the general result covers c_k; dilution is the tractable case where J_k and a_k come along for free.

[24] The analogous forces act on J_k through a parallel signed-kernel formula, but a ranking of J_k requires additional conditions because $J_k(F) = A_k(x_k^\star(F))$ also depends on the movement of the minimizer (see Online Appendix OA.7). This is consistent with the counterexample in Online Appendix OA.4, where J_k increases despite the more dispersed profile having greater convex order.

5.3 The role of the participation threshold

We close this section by examining how the critical cutoff $c_k(F)$ and the jump size $J_k(F)$ depend on the participation threshold k. First, the cutoff: intuitively, when each agent needs more participating friends, the network must be more connected to sustain participation at all. This increases the cutoff.

Proposition 2. *For any interaction profile F, the critical cutoff $c_k(F)$ is strictly increasing in k.*

Further, the cutoff appears to be approximately linear in k; see Figure 4 below.[25]

In contrast, the size of the jump need not always be increasing in k. However, numerical exercises show that for natural parametric families, the size of the jump is increasing and concave. For example, in the homogeneous benchmark, $J_3(\delta_1) \approx 0.27$, $J_{10}(\delta_1) \approx 0.74$, and $J_{100}(\delta_1) \approx 0.95$. Numerically, $J_k(\delta_1)$ saturates toward 1 at high k. Heterogeneous profiles we plot exhibit the same monotonicity but generally at lower levels, with the apparent saturating value below 1.

Intuitively, at the cutoff, the typical agent has comfortably more than k friends—and this margin widens as k grows. A higher participation threshold raises the bar for the marginal agent, but the average agent exceeds the threshold by an even larger margin; the k-core that emerges at the cutoff therefore covers a larger share of the population.

Figure 4 plots both objects, c_k and J_k, in the homogeneous benchmark, the gamma family, and the core-periphery environment up to $k = 150$. Taking $k = 150$ as the upper limit is a deliberate choice: it is a common benchmark for the size of an individual's typical full social network (Dunbar, 1998). The empirically relevant range for behaviors driven by close peer reinforcement is considerably smaller: representative U.S. survey data put the average number of close confidants at approximately two (McPherson et al., 2006), and experimental work on complex contagion finds effective thresholds in the single digits (Centola, 2010).[26]

[25] In the homogeneous case, this can be shown formally for sufficiently large k. Pittel et al. (1996) showed that $c_k(\delta_1) = k + \sqrt{k \log k} + O(\log k)$ as $k \to \infty$. The $\sqrt{k \log k}$ term becomes negligible as $k \to \infty$, so $c_k(\delta_1) \sim k$.

[26] The Bernoulli dilution case is omitted from the figure: Remark 3 provides exact scaling identities $c_k(F^{(\nu)}) = \nu\, c_k(F)$ and $J_k(F^{(\nu)}) = \nu\, J_k(F)$, so the shape in k under dilution inherits directly from the underlying profile F.

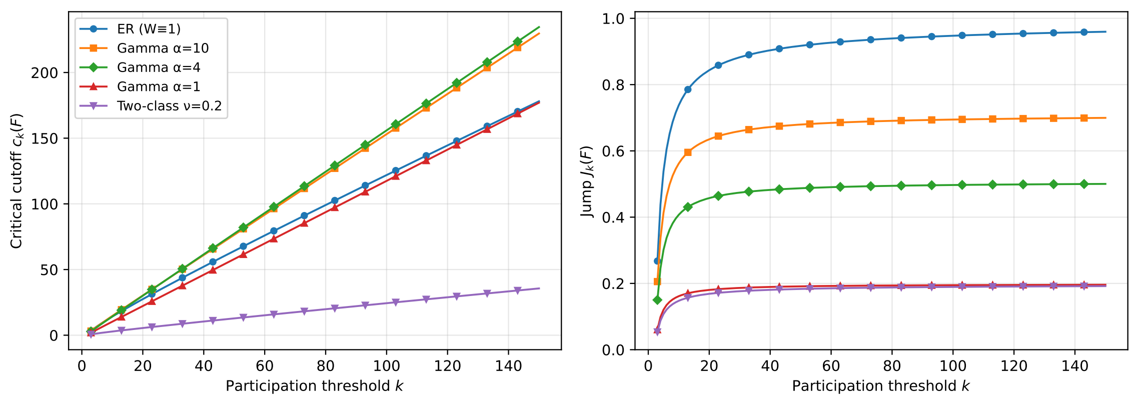

Figure 4: Critical cutoff $c_k(F)$ (left panel) and jump size $J_k(F)$ (right panel) as functions of the participation threshold k, for the homogeneous benchmark, the gamma family, and the core-periphery (two-class) environment. The cutoff rises monotonically and approximately linearly in k; the jump rises monotonically with concave shape, saturating toward 1 in the homogeneous benchmark and at lower, profile-dependent values for the heterogeneous families.

The two plots carry distinct economic implications for the principal's design problem. A higher k pushes the cutoff up—giving the principal more room to raise r while still enjoying zero participation—and it amplifies the jump, making movement just above the cutoff more costly. Together, these make the zero-participation regime more attractive at higher k. Pulling the other way, Theorem 2(ii) implies that within an interior regime, a higher k reduces participation at any fixed r, which all else equal pushes the principal toward a higher interaction rate. Section 6 examines the resulting trade-off and shows that the principal's optimal interaction rate can move in either direction in response to a change in k. We now turn to the principal's problem.

6 How the Principal Designs the Network

The principal's problem is to choose the interaction rate r to maximize her payoff (1), taking as given how agents will behave in the second stage—which is in turn a function of the participation threshold, k, and the interaction profile, F. Substituting the equilibrium participation $\bar{a} = a_k(r; F)$ from Section 4, she maximizes

$$\pi(r) = \alpha r - \tfrac{1}{2}\, r^2 - \beta\, a_k(r; F). \tag{11}$$

The headline result is that if the interaction profile is homogeneous, the principal will never choose an interaction rate that is close to but above the critical cutoff, $c_k(F)$. This is precisely due to the large jump in the level of participation (i.e. the discontinuous emergence of the giant k-core) at the critical cutoff—the qualitative results do not depend on the particular functional form of the principal's payoffs. Setting the interaction rate exactly at the cutoff delivers no participation at all. But going even a very small amount above the cutoff suddenly yields significant participation. This creates significant costs for the principal, but only small incremental benefits. So she is better off by either "returning" to the critical cutoff, or "pushing through" to a much higher interaction rate—generating large enough direct benefits from interactions to offset the participation costs.

We call this push to extreme choices the "missing middle". But note that the missing middle is only economically meaningful when the jump is large. In particular, highly heterogeneous interaction profiles can generate small jumps at the critical cutoff, and hence small losses from choosing an interaction rate just above that cutoff.

We also examine how changes in parameters alter the principal's optimal choice. A higher cost to participation, β, leads the principal to choose a lower interaction rate, and can induce a switch to complete suppression of participation. In contrast, an increase in the participation threshold, k, can go either way. While it reduces participation for any fixed interaction rate (Theorem 2(ii)), the principal may respond by increasing the interaction rate by so much that it overwhelms this effect.

6.1 Choosing the optimal interaction rate

Preliminary observations. To start, let $r^{\text{naive}} = \alpha$. This is what the principal would choose absent any concerns about participation (i.e. $\beta = 0$). And because the level of participation, $a_k(r; F)$, is weakly increasing in the interaction rate, the principal never chooses an interaction rate above this level (i.e. $r > r^{\text{naive}}$). Next, note that since π is upper semicontinuous and non-negative on a compact interval (due to the convex costs), it attains its maximum on that interval.Moreover, the optimum is generically unique. This is because the value function $V(\beta) \equiv \sup_{r \geq 0} \pi(r; F)$ is convex in β (as the pointwise supremum of affine functions), hence differentiable except at a countable set. At any β where V is differentiable, the global maximizer is unique (we prove this in Online Appendix OA.8.8). So if there are two or more optimal choices of the interaction rate then a small perturbation of β restores uniqueness.

Finally, notice that the problem is trivial if $\alpha \le c_k(F)$. In this case, the principal's unconstrained optimum, $r^{\text{naive}} = \alpha$, lies *below* the critical cutoff and triggers no participation. So throughout this section we focus on the case where $\alpha > c_k(F)$. This is where the principal faces a genuine tension between the benefits of interaction and the costs of the participation it induces.

The missing middle. The discontinuity in equilibrium participation creates a distinctive feature of the principal's problem. Because participation jumps discretely from zero to at least $J_k(F)$ when r crosses the critical cutoff $c_k(F)$, the principal will never choose an interaction rate "just above" the cutoff. If she moves even slightly beyond $c_k(F)$, she incurs a participation cost of at least $\beta J_k(F)$—a discrete penalty—while the direct benefit $\alpha r - \frac{1}{2}r^2$ improves only marginally relative to its value at $c_k(F)$. For each $\beta \ge 0$, define

$$\pi_\beta(r;F) \equiv \alpha r - \frac{1}{2}r^2 - \beta\, a_k(r;F), \qquad f(\beta;\alpha,F) \equiv \sup_{r > c_k(F)} \pi_\beta(r;F).$$

Then

$$\bar{\beta}(\alpha;F) = \sup\big\{\beta > 0 : f(\beta;\alpha,F) \ge \pi_\beta\big(c_k(F);F\big)\big\}\,. \tag{12}$$

With this notation in place, we can now formalize the "missing middle" intuition.

Proposition 3. *Suppose $\alpha > c_k(F)$, and let $\bar{\beta}(\alpha;F)$ be defined in* (12).

(i) (High participation costs.) *if $\beta > \bar{\beta}$, the optimal interaction rate is exactly at the critical cutoff (and the level of participation is zero): $r^* = c_k(F)$.*

(ii) (Low participation costs.) *if $\beta < \bar{\beta}$, the optimal interaction rate is at least a distance $d_k(F) > 0$ above the critical cutoff: $r^* \ge c_k(F) + d_k(F)$, where*

$$d_k(F) = \frac{\beta\, J_k(F)}{\alpha - c_k(F)} > 0. \tag{13}$$

The result splits optimal behavior into two regimes, with a sharp boundary at $\bar{\beta}$. When participation is sufficiently costly for the principal, she will choose the highest interaction rate that yields no participation. And when participation is not too costly, she will choose an interaction rate significantly above the critical cutoff, accepting a high level of participation in the process.[27]

The term $d_k(F)$ provides a lower bound on how far the principal must move up from the critical cutoff to recover her losses. It of course depends on the parameters α and β. But its dependence on the rest of the environment is entirely through the critical cutoff $c_k(F)$ and the jump $J_k(F)$—the two key summary statistics from Section 4. The comparison theorem (Theorem 4) clarifies which forms of heterogeneity move these objects: heterogeneity weakens the missing middle when it creates more agents near the margin of self-sustaining participation, rather than concentrating interaction among agents already safely inside the core. A lower c_k, combined with a smaller J_k, reduces $d_k(F)$ and narrows the missing middle.

Heterogeneity in interactions therefore determines the size of the missing middle. In homogeneous environments, $J_k(F)$ is large, so the missing middle is large. In contrast, more heterogeneous environments have a small jump, which weakens the discrete tradeoff near the cutoff and makes an interaction rate closer to the critical cutoff less costly.[28]

The missing middle also creates a sensitivity in the principal's optimal choice of r. Although any equilibrium with positive participation has *a lot* of participation, small changes in the incentives (α or β) may induce the principal to "flip" from an equilibrium with substantial participation to one with zero (or vice versa). To explore this sensitivity, we now describe how the principal's equilibrium choice of r depends on the participation cost β.

6.2 Comparative statics and tipping points

The next result gives both a robust (set-valued) comparative static and a stronger derivative statement at regular interior optima. Let $R(\beta) \equiv \arg\max_{r \geq 0} \pi_\beta(r; F)$ denote the set of optimal interaction rates, and let $\bar{a}^* \equiv a_k(r^*; F)$ denote the equilibrium

[27]The cutoff $\bar{\beta}(\alpha; F)$ is a knife-edge in the middle, where there is both an optimal choice at the critical cutoff and one significantly above it. The principal is of course indifferent between these choices.

[28]Of course, the global optimum may lie well above $c_k(F_\nu)$. This will depend on the parameters (α, β). Proposition 3 characterizes a region above the cutoff $c_k(F)$ that can *never* be optimal. Additionally, note that our analysis has focused on the case where there is only one jump in the size of the giant k-core (this is the case where the function g has a unique minimizer), and hence in the level of participation. If participation has additional jumps at higher r, an analogous missing middle can arise there; we leave that extension for future work.

participation at the optimal policy.

Proposition 4 (Comparative statics and tipping points)**.** *Fix $\alpha > c_k(F)$ and let $\bar{\beta} = \bar{\beta}(\alpha; F)$.*

(i) (Monotonicity.) *If $0 \le \beta_1 < \beta_2$ and $r_i \in R(\beta_i)$, then $r_2 \le r_1$.*

(ii) (Strict monotonicity.) *Moreover, if $0 \le \beta_1 < \beta_2 < \bar{\beta}$ and each $R(\beta_i)$ is a singleton $\{r_i\}$, then $r_2 < r_1$ and $a_k(r_2; F) < a_k(r_1; F)$.*

(iii) (Local derivative.) *At any $\beta < \bar{\beta}$ where the maximizer is unique, interior, and nondegenerate,*

$$\frac{\mathrm{d}r^*}{\mathrm{d}\beta} < 0 \quad \textit{and} \quad \frac{\mathrm{d}\bar{a}^*}{\mathrm{d}\beta} < 0.$$

Proposition 4 shows the difference between the two regimes. When the principal tolerates the behavior in equilibrium ($\beta < \bar{\beta}$), adjustments are smooth: as β increases, the principal reduces the interaction rate. This, in turn, leads to a reduction in participation. In contrast, when the optimal strategy is to completely suppress participation ($\beta > \bar{\beta}$), the principal's choice of r^* is "sticky". Small changes in her incentives are insufficient to move the interaction rate away from the critical cutoff; she continues to choose the highest possible interaction rate that guarantees zero participation.

The key takeaway from this result is that when β is close to $\bar{\beta}$, a small change in the cost of participation can move her from choosing an equilibrium with positive participation to one with zero participation (or the reverse). This highlights a tipping-point phenomenon in the principal's decision-making: her strategy, and more importantly the resulting participation, can shift dramatically in response to relatively mild changes in her incentives.

Raising the threshold. A higher participation threshold k exerts two opposing forces on the principal's choice. For any fixed $r > c_k$, it decreases the participation $a_k(r; F)$ (Theorem 2(ii)), pushing the principal toward higher r; but it also raises c_k and—for the interaction environments we plot in Figure 4—the size of the jump $J_k(F)$, making it more attractive to fully suppress participation. In the cases we have examined numerically, equilibrium participation $\bar{a}^*$ is always weakly decreasing in k, while the optimal interaction rate r^* can move in either direction: Online Appendix OA.6

provides examples that go both ways. We leave a sharper characterization for future work.

7 Concluding Remarks

We finish by briefly commenting on some of our modeling choices and how they relate to directions for future research.

Strategic complementarities. The model's core mechanism is a participation threshold driven by strategic complementarities. We interpret this broadly as a process of social reinforcement, where an agent's incentive to act is fundamentally linked to the actions of his direct network connections. The parameter k represents the critical mass of local support required to make an action worth taking. In our setting, this threshold is an *absolute number* of neighbors, rather than a fraction of all neighbors. This is technically useful, as it allows us to build on existing work in random graph theory. And it corresponds to settings where agents need a certain "critical mass" of local support for an action—such as participating in a protest, where safety in numbers requires enough fellow participants regardless of total network size, or adopting a new norm that requires a minimum number of compatible peers. Settings where agents instead care about the *share* of neighbors taking an action—such as language choice, where speakers switch when most of their contacts have adopted the new language—would benefit from a different modeling approach.

The random network. Formally, our random network is a mixed-Poisson model.[29] In the configuration model, agents form links uniformly at random. This is a deliberate assumption to isolate the interplay between complex contagion and network heterogeneity in expected degrees (i.e. agents' propensities to form links). However, our model abstracts from several important features of real-world networks. These include homophily—the tendency for agents to form links with similar others—and multiplexity, the idea that different types of links can play different roles. Additionally, agents in our framework do not make endogenous decisions about forming or severing links; the network is determined by the principal's choice of interaction rate

[29] This nests the canonical Erdős–Rényi network as a special case when $W \equiv 1$. When $W \equiv 1$, agents' degrees are i.i.d. Poisson(r), which is asymptotically equivalent to $G(n, r/n)$.

and the realization of the random graph. In reality, agents may have some control over both how many connections to form and with whom to form them.

Relaxing these assumptions provides a clear agenda for future work. Incorporating homophily into the configuration model—for instance through a stochastic block model with heterogeneous degrees—would allow the analysis to capture group-level variation in both connectivity and behavior. Endogenous link formation, where agents can respond to the principal's choice by rewiring their connections, represents a more significant challenge but is important for applications where agents strategically manage their exposure to contagion.

The principal's levers. In our setting, the principal's policy levers are summarized by a single parameter, the interaction rate r. In practice, this may reflect a bundle of decisions—such as the frequency of meetings, the layout of shared spaces, or the length of breaks—that collectively shape how much agents interact. But we are considering a setting where the principal lacks more fine-grained tools required to target specific groups of agents, or induce agents to interact with specific people. Hence she treats the interaction profile F as fixed. A natural extension is a joint dynamic problem in which policy choices can also shape F; we leave that extension for future work.

Another natural tool for the principal is targeted interventions—such as degree-dependent subsidies or selective seeding of agents to shift their behavior. In our framework, random seeding decomposes cleanly via Poisson thinning, but the interaction between seeding and endogenous connectivity choice is richer: when the principal can respond to seeding by adjusting r, the net effect on participation may depend on the interaction profile in ways that our analysis of the missing middle suggests but does not fully resolve. We view this as another promising direction for future work.

An advantage for empirical work. Despite the abstractions, our framework's tractability offers a practical advantage: the key tradeoffs are driven by the interaction rate r and the interaction profile F, rather than by the full network adjacency matrix. This makes the model's predictions testable with aggregate data on connectivity patterns, and should serve as a useful benchmark for richer models that incorporate more detailed network features.

Moreover, the heterogeneity in our interaction profile F is directly tied to degree moments. The variance in realized degrees depends only on the interaction rate r and the variance in the weights W_i. Formally: $\text{Var}(D_i) = \mathbb{E}[\text{Var}(D_i \mid W_i)] + \text{Var}(\mathbb{E}[D_i \mid W_i]) = r + r^2\text{Var}(W_i)$. So the variance in our (unobservable) interaction profile F is identified by the mean and variance of the realized degree distribution.

Wrapping up. This paper sets out a tractable framework for analyzing complex contagions on heterogeneous networks. By using a random networks approach we are able to provide clean comparative statics for how aspects of network structure affect equilibrium behavior. Notably, we highlight that a sudden jump in behavior around a critical level of network connectivity arises directly from the threshold-based nature of complex contagion, and that heterogeneity in the network structure influences the size of this jump.

These results then unlock an analysis of a principal's optimal design problem, where she can choose network connectivity but not the finer structure of the network. When the network structure is homogeneous, the principal is forced into extreme all-or-nothing policies. She either suppresses connectivity to prevent any participation or sets connectivity well above the critical cutoff, allowing significant participation. There is no middle ground. This is driven by the jump. If the principal accepts any participation, she must accept significant participation, so she is "in for a penny, in for a pound."

But in highly heterogeneous environments—such as those with a core–periphery structure—more moderate policies can arise. This is precisely because the jump in participation at the cutoff is smaller, which weakens the force driving the extreme policies. Together, these insights provide a new link between network heterogeneity and the qualitative character of optimal policy, and help explain why principals in some environments resort to extreme measures (such as internet shutdowns) while in others, the pressure toward extremes is weaker and moderate policies can emerge.

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A Appendix A: Proofs

A.1 Proof of Theorem 1 (Participation)

Proof. Let D_i denote the degree of agent i. Conditional on W_i, $D_i \mid W_i \sim \text{Poisson}(rW_i)$ independently across i. Unconditionally, the D_i are i.i.d. with mixed-Poisson law

$$p_j(r;F) \equiv \mathbb{P}(D_i = j) = \mathbb{E}\left[e^{-rW}\frac{(rW)^j}{j!}\right], \qquad j \geq 0,$$

and mean $\mathbb{E}[D_i] = r$. Let $n_j^{(n)} \equiv \#\{i \leq n : D_i = j\}$. By the strong law of large numbers,

$$\frac{n_j^{(n)}}{n} \to p_j(r;F) \qquad \text{for every } j \geq 0, \qquad \frac{1}{n}\sum_{i=1}^{n} D_i \to r$$

almost surely (the joint statement across j following from countable intersection of probability-one events). Thus the realized degree sequence satisfies Condition 2.1 of Janson (2009) almost surely. Since Proposition 4.1 of Janson (2009) is stated conditional on the realized degree sequence, its conclusions apply to the multigraph configuration model on this almost-sure event. Under Assumption 1, the configuration model is simple with probability bounded away from zero, so the convergence transfers to the simple-graph-conditioned model we use in Section 3.

Let D denote a random variable with the limiting degree law $(p_j(r;F))_{j\geq 0}$, and let D_p denote its p-thinning (i.e. $D_p \sim \text{Bin}(D,p)$). Conditional on W, $D_p \mid W \sim \text{Poisson}(rWp)$. Therefore $h_1(p) \equiv \mathbb{P}(D_p \geq k) = \mathbb{E}[\psi_k(rWp)] = A_k(rp)$ (see (3)). Similarly, the standard Poisson identity $\mathbb{E}[N\,\mathbf{1}_{\{N\geq k\}}] = \lambda\,\psi_{k-1}(\lambda)$ for $N \sim \text{Poisson}(\lambda)$, applied conditional on W, gives $\mathbb{E}[D_p\,\mathbf{1}_{\{D_p\geq k\}} \mid W] = rWp\,\psi_{k-1}(rWp)$. Taking expectations,

$$\begin{aligned} h(p) &\equiv \mathbb{E}[D_p \, \mathbf{1}_{\{D_p \geq k\}}] \\ &= \mathbb{E}[rWp \, \psi_{k-1}(rWp)] \\ &= rp \, \Phi_k(rp). \end{aligned}$$

The root equation in Janson's (2009) Proposition 4.1, in our notation, is $h(p) = rp^2$, i.e. $rp \, \Phi_k(rp) = rp^2$. For $p = 0$ this is trivial; for $p > 0$, setting $x = rp$ gives $x = r\Phi_k(x)$. Thus positive roots correspond exactly to positive solutions of the fixed-point equation $x = r\Phi_k(x)$, and for such a solution $h_1(p) = A_k(x)$ with $p = x/r$.

If $r < c_k(F)$, then $x > r\Phi_k(x)$ for every $x > 0$, so there are no positive roots. Proposition 4.1(i) of Janson (2009) then gives $v(\text{Core}_k)/n \xrightarrow{p} 0$.

Now suppose $r > c_k(F)$. By Assumption 2, the function $g(x) = x/\Phi_k(x)$ has a unique minimizer $x_k^\star(F)$ and is strictly increasing on $[x_k^\star(F), \infty)$. Let $x_k(r)$ denote the largest positive solution to $x = r\Phi_k(x)$. Then $x_k(r) > x_k^\star(F)$. Choose $\delta \in (0, x_k(r) - x_k^\star(F))$. For every $x \in (x_k(r) - \delta, x_k(r))$, $g' > 0$ implies $g(x) < g(x_k(r)) = r$, i.e. $x < r\Phi_k(x)$. Writing $p = x/r$, we obtain

$$h(p) - rp^2 = rp\big(\Phi_k(rp) - p\big) > 0 \qquad \text{for all } p \in \big((x_k(r) - \delta)/r, \, x_k(r)/r\big).$$

By Janson (2009, Remark 4.3), this is exactly the non-local-maximum condition required in Proposition 4.1(ii). Therefore

$$\frac{v(\text{Core}_k)}{n} \xrightarrow{p} A_k(x_k(r)) > 0.$$

By Remark 1, in every realized network the fraction of participating agents equals $v(\text{Core}_k)/n$. Hence the above convergence is exactly the law of large numbers for equilibrium participation, giving (6) for $r > c_k(F)$.

At the exact critical cutoff $r = c_k(F)$, Proposition 4.1 of Janson (2009) does not apply. We make no claim about the finite-n k-core at this point. We adopt the convention $a_k(c_k(F); F) = 0$ which is without loss for our purposes—see footnote 18. □

A.2 Proof of Proposition 1 (Jump size)

Proof. By Assumption 2, g has a unique minimizer $x_k^\star(F)$ and is strictly increasing on $[x_k^\star(F), \infty)$. For each $r > c_k(F) = g(x_k^\star(F))$, the equation $r = g(x)$ has a unique solution on $[x_k^\star(F), \infty)$; this solution is the largest fixed point, which we denote by $x_k(r)$. Since g is continuous (by dominated convergence) and strictly increasing on $[x_k^\star(F), \infty)$, its inverse on that interval is continuous, so $x_k(r) \downarrow x_k^\star(F)$ as $r \downarrow c_k(F)$. Because $a_k(r; F) = A_k(x_k(r))$ and A_k is continuous,

$$J_k(F) = \lim_{r \downarrow c_k(F)} a_k(r; F) = A_k(x_k^\star(F)).$$

Moreover, this value is strictly positive: since $x_k^\star(F) > 0$, $\mathbb{P}(W > 0) > 0$, and $\psi_k(y) > 0$ for every $y > 0$, the expectation $A_k(x_k^\star(F)) = \mathbb{E}[\psi_k(W x_k^\star(F))]$ is strictly positive. Hence $J_k(F) > 0$. □

A.3 Proof of Theorem 2 (Comparative statics)

Proof. (i) Monotonicity in the interaction rate. Fix the participation threshold k, and let $x_k(r)$ denote the largest solution to $x = r\Phi_k(x)$. By Theorem 1, $a_k(r; F) = A_k(x_k(r))$, and A_k is strictly increasing on $(0, \infty)$ because $\mathbb{P}(W > 0) > 0$ and ψ_k is strictly increasing on $(0, \infty)$.

For $r \le c_k(F)$, Theorem 1 gives $a_k(r; F) = 0$, so weak monotonicity is immediate whenever $r_2 \le c_k(F)$ or $r_1 \le c_k(F) < r_2$. It remains to establish strict monotonicity for $r_2 > r_1 > c_k(F)$.

Let $r_2 > r_1 > c_k(F)$, and write $T_r(x) \equiv r\Phi_k(x)$. The map T_{r_2} is continuous and increasing in x, with $T_{r_2}(x) = r_2\Phi_k(x) \le r_2$ for every $x \ge 0$ (since $\Phi_k(x) \le \mathbb{E}[W] = 1$); so T_{r_2} maps $[0, \infty)$ into $[0, r_2]$. Because $x_k(r_1) > 0$ and $\Phi_k(x_k(r_1)) > 0$ (the latter from $\mathbb{P}(W > 0) > 0$ and $\psi_{k-1}(y) > 0$ for every $y > 0$),

$$T_{r_2}(x_k(r_1)) = r_2\Phi_k(x_k(r_1)) > r_1\Phi_k(x_k(r_1)) = x_k(r_1). \tag{14}$$

Define the iteration $y_0 \equiv x_k(r_1)$, $y_{m+1} \equiv T_{r_2}(y_m)$. By (14), $y_1 > y_0$. Since T_{r_2} is increasing, applying it preserves this inequality at every step, so (y_m) is non-decreasing. It is bounded above by r_2, so it converges to some limit L, and continuity of T_{r_2} gives $T_{r_2}(L) = L$—i.e., L is a fixed point of T_{r_2}. Because $y_m \ge y_1$ for every $m \ge 1$, the strict gap in (14) survives in the limit: $L \ge y_1 > x_k(r_1)$. Since $x_k(r_2)$ is the *largest* fixed point of T_{r_2}, $x_k(r_2) \geq L > x_k(r_1)$. Strict monotonicity of A_k then gives $a_k(r_2; F) > a_k(r_1; F)$.

(ii) Monotonicity in the participation threshold. Fix the interaction rate r, and write $T_{k,r}(x) \equiv r\Phi_k(x)$. For every $x > 0$,

$$\Phi_{k+1}(x) = \mathbb{E}[W\psi_k(Wx)] < \mathbb{E}[W\psi_{k-1}(Wx)] = \Phi_k(x),$$

because $\mathbb{P}(W > 0) > 0$ and $\psi_k(y) < \psi_{k-1}(y)$ for every $y > 0$. Therefore $T_{k+1,r}(x) \leq T_{k,r}(x)$ for all $x \geq 0$. If $x_{k+1}(r) = 0$ then $x_{k+1}(r) \leq x_k(r)$ trivially; otherwise $T_{k,r}(x_{k+1}(r)) \geq T_{k+1,r}(x_{k+1}(r)) = x_{k+1}(r)$, so iterating $T_{k,r}$ from $x_{k+1}(r)$ produces a non-decreasing sequence bounded above by r, which converges to a fixed point of $T_{k,r}$ at least $x_{k+1}(r)$. Since $x_k(r)$ is the largest such fixed point, $x_k(r) \geq x_{k+1}(r)$ in either case. Using again the representation from Theorem 1,

$$\begin{aligned} a_{k+1}(r; F) &= \mathbb{E}[\psi_{k+1}(Wx_{k+1}(r))] \\ &\leq \mathbb{E}[\psi_k(Wx_{k+1}(r))] \\ &\leq \mathbb{E}[\psi_k(Wx_k(r))] \\ &= a_k(r; F). \end{aligned}$$

This proves weak monotonicity in k.

If $a_{k+1}(r; F) > 0$, then necessarily $x_{k+1}(r) > 0$ because $A_{k+1}(0) = 0$. Since $\mathbb{P}(W > 0) > 0$ and $\psi_{k+1}(y) < \psi_k(y)$ for every $y > 0$, the first inequality above is then strict, and hence $a_{k+1}(r; F) < a_k(r; F)$. This is exactly the strict inequality stated in Theorem 2(ii). The remaining case $a_{k+1}(r; F) = 0$ is immediate: when $a_k(r; F) > 0 = a_{k+1}(r; F)$, the strict inequality $a_{k+1}(r; F) < a_k(r; F)$ holds trivially. □

A.4 Proof of Theorem 3 (Jump size and near-homogeneous stability)

Proof. Throughout this proof, write $g^F(x) \equiv x/\Phi_k(x)$ when we want to display the dependence on the interaction profile F, and abbreviate $g^{ER}(x) \equiv x/\psi_{k-1}(x)$ for the Erdős–Rényi case (where $W \equiv 1$ gives $\Phi_k = \psi_{k-1}$). Fix $k \geq 3$ and let $x^\star \equiv x_k^{\star,ER}$ denote the unique minimizer of g^{ER}, with minimum value $c^{ER} \equiv c_k^{ER}$.

Part (i): the homogeneous benchmark. When $W \equiv 1$, $\Phi_k(x) = \psi_{k-1}(x)$, $A_k(x) = \psi_k(x)$, and the minimizer $x^\star$ satisfies the first-order condition $\psi_{k-1}(x^\star) = x^\star \psi'_{k-1}(x^\star)$. Using the standard Poisson identities $\psi'_m(x) = p_{m-1}(x)$, where $p_m(x) \equiv \mathbb{P}(\text{Poisson}(x) = m)$, and $\psi_{k-1} = \psi_k + p_{k-1}$, we obtain

$$\begin{aligned} J_k^{ER} &= \psi_k(x^\star) = \psi_{k-1}(x^\star) - p_{k-1}(x^\star) = x^\star p_{k-2}(x^\star) - p_{k-1}(x^\star) \\ &= (k-1)\, p_{k-1}(x^\star) - p_{k-1}(x^\star) = (k-2)\, p_{k-1}(x^\star), \end{aligned}$$

where the penultimate step uses the Poisson recursion $x\, p_{k-2}(x) = (k-1)\, p_{k-1}(x)$. For $k = 3$, $x^\star$ solves $e^x = 1 + x + x^2$, and $J_3^{ER} = p_2(x^\star) \approx 0.268$.

Part (ii): limited heterogeneity. *Strategy.* We organize the proof in two parts, sharing common setup. *Setup* (Steps 1 and 2): Step 1 confines the minimizer of g^F to a fixed compact interval $[a, b]$ around $x^\star$ that does not depend on F, turning the rest of the argument into a uniform-approximation problem on a compact set; Step 2 shows $g^F \to g^{ER}$ uniformly on $[a, b]$ as $\sigma(F) \to 0$. *Part A* (Step 3) proves cutoff continuity, $|c_k(F) - c^{ER}| \to 0$. *Part B* invokes Lemma 6 from Online Appendix OA.8.5, which verifies Assumption 2 for sufficiently homogeneous profiles, and then combines compactness with uniform approximation of A_k to prove continuity of the jump, $|J_k(F) - J_k^{ER}| \to 0$.

Step 1: compactness of the minimization problem near ER. Fix $\eta > 0$. Because $g^{ER}(x) \to \infty$ as $x \downarrow 0$ and as $x \to \infty$, there exist $0 < a < x^\star < b < \infty$ such that

$$\inf_{x \in (0,a] \cup [b,\infty)} g^{ER}(x) \;\geq\; c^{ER} + 3\eta. \tag{15}$$

We will show that for F sufficiently close to δ_1, the minimizer of g^F must lie in $[a, b]$ as well.

Fix b as above. Note that for every F, $\Phi_k(x) \leq \mathbb{E}[W] = 1$ (since $\psi_{k-1} \leq 1$), so $g^F(x) = x/\Phi_k(x) \geq x$ for all $x \geq 0$. Thus, for the chosen b, $g^F(x) \geq x \geq b$ for all $x \geq b$. In particular, if $b \geq c^{ER} + 3\eta$ (we may enlarge b to ensure this), then

$$\inf_{x \geq b} g^F(x) \;\geq\; c^{ER} + 3\eta \qquad \text{for all } F. \tag{16}$$

Next, fix $a > 0$ as in (15) and (if needed) shrink it further so that $a \leq 1/2$. By

Lemma 4 in Online Appendix OA.8.3, for any interaction profile F with dispersion $\sigma(F) = \sigma$ and any $x \in (0, a]$, $g^F(x) \geq 1/(4x+6\sigma^2) \geq 1/(4a+6\sigma^2)$. Choose $\varepsilon_1 \in (0, 1]$ small enough (and, if needed, shrink a once more) so that $1/(4a + 6\varepsilon_1^2) \geq c^{ER} + 3\eta$. Then for all F with $\sigma(F) \leq \varepsilon_1$,

$$\inf_{x \in (0,a]} g^F(x) \;\geq\; c^{ER} + 3\eta. \tag{17}$$

Combining (16) and (17), for all F with $\sigma(F) \leq \varepsilon_1$: if the global minimum of g^F is below $c^{ER} + 3\eta$, then every minimizer must lie in the compact interval $[a, b]$. We now show that the minimum is indeed achieved below this level.

Step 2: uniform closeness of g^F to g^{ER} on $[a, b]$. On $[a, b]$, ψ_{k-1} is bounded below by $m \equiv \min_{x\in[a,b]} \psi_{k-1}(x) > 0$. By Lemma 5 in Online Appendix OA.8.4, $\sup_{x\in[a,b]} |\Phi_k(x) - \psi_{k-1}(x)| \to 0$ as $\sigma(F) \to 0$. In particular, there exists $\varepsilon_2 > 0$ such that $\sigma(F) \leq \varepsilon_2$ implies $\sup_{x\in[a,b]} |\Phi_k(x)-\psi_{k-1}(x)| \leq m/2$. By the triangle inequality, $\Phi_k(x) \geq \psi_{k-1}(x) - |\Phi_k(x) - \psi_{k-1}(x)| \geq m - m/2 = m/2$ for all $x \in [a, b]$. Therefore, for such F and all $x \in [a, b]$,

$$|g^F(x)-g^{ER}(x)| = \left|\frac{x}{\Phi_k(x)} - \frac{x}{\psi_{k-1}(x)}\right| = x\cdot\frac{|\Phi_k(x) - \psi_{k-1}(x)|}{\Phi_k(x)\psi_{k-1}(x)} \leq b\cdot\frac{\sup_{[a,b]} |\Phi_k - \psi_{k-1}|}{(m/2)\cdot m}.$$

The denominator $(m/2) \cdot m$ is a fixed positive constant; the numerator $\sup_{[a,b]} |\Phi_k - \psi_{k-1}| \to 0$ as $\sigma(F) \to 0$ by Lemma 5. Hence $\sup_{x\in[a,b]} |g^F(x) - g^{ER}(x)| \to 0$ as $\sigma(F) \to 0$.

Part A (Step 3): cutoff continuity. Let $c(F) \equiv c_k(F) = \inf_{x>0} g^F(x)$. Since $x^\star \in [a, b]$, Step 2 gives

$$c(F) \leq g^F(x^\star) \leq g^{ER}(x^\star) + \sup_{[a,b]} |g^F - g^{ER}| = c^{ER} + o(1) \quad \text{as } \sigma(F) \to 0.$$

For $\sigma(F)$ small enough that the $o(1)$ term is below 3η (and $\sigma(F) \leq \varepsilon_1$), we have $c(F) < c^{ER} + 3\eta$, so by Step 1 the infimum is achieved at some $x^F \in [a, b]$.

For the matching lower bound,

$$c^{ER} = g^{ER}(x^\star) \leq g^{ER}(x^F) \leq g^F(x^F) + \sup_{[a,b]} |g^F - g^{ER}| = c(F) + o(1).$$

Combining, $|c(F) - c^{ER}| \to 0$ as $\sigma(F) \to 0$, proving the cutoff continuity claim.

Part B: continuity of the jump. By Lemma 6 in Online Appendix OA.8.5, for $\sigma(F)$ sufficiently small, g^F has a unique minimizer $x_k^\star(F)$ and $J_k(F) = A_k(x_k^\star(F))$ is well-defined. By Step 1, $x_k^\star(F) \in [a,b]$ for $\sigma(F)$ small.

Convergence of the minimizer. Suppose, for contradiction, that there is a sequence $\sigma(F_n) \to 0$ with $x_k^\star(F_n) \not\to x^\star$. By compactness, some subsequence $x_k^\star(F_{n_j}) \to x' \in [a,b]$ with $x' \neq x^\star$. Step 2 gives $g^{F_{n_j}}(x_k^\star(F_{n_j})) - g^{ER}(x_k^\star(F_{n_j})) \to 0$, and continuity of g^{ER} gives $g^{ER}(x_k^\star(F_{n_j})) \to g^{ER}(x')$. By Part A, $g^{F_{n_j}}(x_k^\star(F_{n_j})) = c_k(F_{n_j}) \to c^{ER}$. Combining, $g^{ER}(x') = c^{ER}$—but $x^\star$ is the unique minimizer of g^{ER}, contradiction. So $x_k^\star(F) \to x^\star$ as $\sigma(F) \to 0$.

Continuity of the jump. By Lemma 5, $\sup_{[a,b]} |A_k(x) - \psi_k(x)| \to 0$ as $\sigma(F) \to 0$. Combined with continuity of ψ_k and $x_k^\star(F) \to x^\star$,

$$|J_k(F) - J_k^{ER}| = |A_k(x_k^\star(F)) - \psi_k(x^\star)| \le \sup_{[a,b]} |A_k - \psi_k| + |\psi_k(x_k^\star(F)) - \psi_k(x^\star)| \to 0,$$

proving (ii). □

A.5 Proof of Remark 3 (Bernoulli dilution scaling)

Proof. Let $W \sim F$ and $B \sim \text{Bernoulli}(\nu)$ be independent and define $W^{(\nu)} \equiv (B/\nu)W$.

Part (i): scaling identities. Using $B \in \{0,1\}$,

$$\Phi_k^{(\nu)}(x) = \mathbb{E}\left[\frac{B}{\nu} W \, \psi_{k-1}\left(\frac{B}{\nu} W x\right)\right] = \nu \cdot \mathbb{E}\left[\frac{1}{\nu} W \, \psi_{k-1}\left(\frac{1}{\nu} W x\right)\right] = \Phi_k(x/\nu).$$

Also $\psi_k(0) = 0$ for $k \ge 1$, so

$$A_k^{(\nu)}(x) = \mathbb{E}\left[\psi_k\left(\frac{B}{\nu} W x\right)\right] = \nu \, \mathbb{E}\left[\psi_k\left(\frac{1}{\nu} W x\right)\right] = \nu \, A_k(x/\nu).$$

Thus $g^{(\nu)}(x) = x/\Phi_k^{(\nu)}(x) = \nu g(x/\nu)$ and taking infima yields $c_k(F^{(\nu)}) = \nu c_k(F)$.

Part (ii): Assumption 2. Suppose F satisfies Assumption 2. From Part (i), $g^{(\nu)}(x) = \nu g(x/\nu)$, so $g^{(\nu)}$ has a unique minimizer at $\nu x_k^\star(F)$. Differentiating,

$$(g^{(\nu)})'(x) = \nu \cdot g'(x/\nu) \cdot \frac{1}{\nu} = g'(x/\nu).$$

For $x > \nu x_k^\star(F)$, $x/\nu > x_k^\star(F)$, so $g'(x/\nu) > 0$ by Assumption 2 applied to F. Hence $(g^{(\nu)})'(x) > 0$ for all $x > \nu x_k^\star(F)$, and $F^{(\nu)}$ also satisfies Assumption 2. Therefore

$$J_k(F^{(\nu)}) = A_k^{(\nu)}(\nu x_k^\star(F)) = \nu A_k(x_k^\star(F)) = \nu J_k(F).$$

The two-class (core–periphery) family F_ν is the special case $F = \delta_1$: setting $W \equiv 1$ above gives $c_k(F_\nu) = \nu\, c_k^{ER}$ and $J_k(F_\nu) = \nu\, J_k^{ER}$. □

A.6 Proof of Proposition 2 ($c_k(F)$ strictly increasing in k)

Proof. The pointwise inequality $\Phi_{k+1}(x) < \Phi_k(x)$ on $\{W > 0\}$ established in the proof of Theorem 2(ii) above gives $x/\Phi_{k+1}(x) > x/\Phi_k(x)$ for any $x > 0$. Let $x_{k+1}^\star$ minimize $g_{k+1}(x) = x/\Phi_{k+1}(x)$. Then

$$c_{k+1}(F) \;=\; \frac{x_{k+1}^\star}{\Phi_{k+1}(x_{k+1}^\star)} \;>\; \frac{x_{k+1}^\star}{\Phi_k(x_{k+1}^\star)} \;\geq\; \inf_{x>0} \frac{x}{\Phi_k(x)} \;=\; c_k(F).$$ □

A.7 Proof of Proposition 3 (The missing middle)

Proof. Let $c \equiv c_k(F)$ and $J \equiv J_k(F) > 0$, and write

$$a(r) \equiv a_k(r;F), \qquad u(r) \equiv \alpha r - \frac{1}{2}r^2, \qquad \pi_\beta(r;F) \equiv u(r) - \beta a(r).$$

For $r \leq c$, Theorem 1 gives $a(r) = 0$, so $\pi_\beta(r;F) = u(r)$. Since $u'(r) = \alpha - r > 0$ on $[0,c]$ (using $\alpha > c$), we have $\pi_\beta(r;F) < u(c) = \pi_\beta(c;F)$ for all $r < c$. By Lemma 7 in Online Appendix OA.8.6, there exists a unique $\bar{\beta} \in (0,\infty)$ such that

$$f(\bar{\beta}) = u(c), \qquad f(\beta) > u(c) \text{ for } \beta < \bar{\beta}, \qquad f(\beta) < u(c) \text{ for } \beta > \bar{\beta},$$

where $f(\beta) \equiv \sup_{r>c} \pi_\beta(r;F)$ as in (12).

Proof of part (i). If $\beta > \bar{\beta}$, then for every $r > c$, $\pi_\beta(r;F) \leq f(\beta) < u(c) = \pi_\beta(c;F)$. Together with the strict inequality $\pi_\beta(r;F) < \pi_\beta(c;F)$ for all $r < c$, this implies that $r = c$ is the unique optimizer.

Proof of part (ii). If $\beta < \bar{\beta}$, then $f(\beta) > u(c) = \pi_\beta(c;F)$, so no optimizer can equal c. As shown above, no optimizer can lie below c either. Hence every optimizer $r^\star$ satisfies $r^\star > c$.

Now fix any optimal $r^\star > c$. Since c is feasible, $\pi_\beta(r^\star; F) \geq \pi_\beta(c; F)$, i.e. $\alpha r^\star - \frac{1}{2}(r^\star)^2 - \beta a(r^\star) \geq \alpha c - \frac{1}{2}c^2$. Using $a(r^\star) \geq J$, a necessary condition is

$$\alpha r^\star - \tfrac{1}{2}(r^\star)^2 - \beta J \geq \alpha c - \tfrac{1}{2}c^2.$$

Writing $r^\star = c+\varepsilon$ with $\varepsilon > 0$ gives $\frac{1}{2}\varepsilon^2-(\alpha-c)\varepsilon+\beta J \leq 0$. Hence ε must be at least the smaller root, $\varepsilon_- = (\alpha-c)-\sqrt{(\alpha-c)^2 - 2\beta J}$. Using $\sqrt{x^2-y} \leq x-y/(2x)$ for $x > 0$, we obtain $\varepsilon_- \geq \beta J/(\alpha-c)$. Therefore $r^\star \geq c + \beta J_k(F)/(\alpha-c) = c_k(F) + d_k(F)$. □

A.8 Proof of Proposition 4 (Comparative statics and tipping points)

Proof. Throughout, write $c \equiv c_k(F)$ and $a(r) \equiv a_k(r; F)$.

Part (i): Monotonicity. Fix $0 \leq \beta_1 < \beta_2$ and choose $r_i \in R(\beta_i)$. Optimality implies

$$\pi_{\beta_1}(r_1; F) \geq \pi_{\beta_1}(r_2; F), \qquad \pi_{\beta_2}(r_2; F) \geq \pi_{\beta_2}(r_1; F).$$

Adding these inequalities and canceling the common terms $\alpha r - \frac{1}{2}r^2$ gives $(\beta_2 - \beta_1)(a(r_2) - a(r_1)) \leq 0$, so $a(r_2) \leq a(r_1)$.

If $r_2 \leq c$, then on $[0, c]$ we have $a(r) = 0$ and $\pi_{\beta_2}(r; F) = \alpha r - \frac{1}{2}r^2$, which is strictly increasing because $\alpha > c$. Hence any optimizer in $[0, c]$ must equal c, so $r_2 = c$. Since c is feasible at β_1, optimality of r_1 gives $\pi_{\beta_1}(r_1; F) \geq \pi_{\beta_1}(c; F)$, and therefore $r_1 \geq c = r_2$.

If instead $r_2 > c$, then Theorem 1(i) gives $a(r_2) > 0$. Since $a(r_2) \leq a(r_1)$, we also have $a(r_1) > 0$, so again by Theorem 1(i) we must have $r_1 > c$. On (c, ∞), the map $a(\cdot)$ is strictly increasing by Theorem 2(i). Therefore $a(r_2) \leq a(r_1)$ implies $r_2 \leq r_1$.

Part (ii): Strict monotonicity. By Lemma 8 in Online Appendix OA.8.7, a is C^2 on (c, ∞) with $a' > 0$. Assume $0 \leq \beta_1 < \beta_2 < \bar{\beta}$ and that each $R(\beta_i)$ is the singleton $\{r_i\}$. By Proposition 3(ii), both optimizers satisfy $r_1, r_2 > c$. Part (i) already gives $r_2 \leq r_1$.

Suppose, for contradiction, that $r_2 = r_1 \equiv \hat{r}$. Then the same interior point $\hat{r} > c$ is optimal for both β_1 and β_2. Differentiability of π_β at $\hat{r}$ means $\hat{r}$ must satisfy the first-order condition

$$0 = \frac{\partial}{\partial r}\pi_\beta(\hat{r}; F) = \alpha - \hat{r} - \beta a'(\hat{r})$$

at both values of β. Since $a'(\hat{r}) > 0$, this equation determines β uniquely at a given $\hat{r}$, a contradiction. Hence $r_2 < r_1$, and because $a(\cdot)$ is strictly increasing on (c, ∞), $a(r_2) < a(r_1)$.

Part (iii): Local derivative. Let $\beta < \bar{\beta}(\alpha; F)$ be such that the maximizer is unique, interior, and nondegenerate, and denote it by $r^*(\beta) > c$. Define

$$H(\beta, r) \equiv \frac{\partial}{\partial r}\pi_\beta(r; F) = \alpha - r - \beta a'(r).$$

Then

$$H(\beta, r^*(\beta)) = 0, \qquad \frac{\partial H}{\partial r}(\beta, r^*(\beta)) = -1 - \beta a''(r^*(\beta)) = \frac{\partial^2}{\partial r^2}\pi_\beta(r^*(\beta); F) < 0.$$

By the implicit function theorem, $r^*(\cdot)$ is differentiable at β, with

$$\frac{\mathrm{d}r^*}{\mathrm{d}\beta} = -\frac{\partial H/\partial \beta}{\partial H/\partial r} = -\frac{-a'(r^*)}{-1 - \beta a''(r^*)} = -\frac{a'(r^*)}{1 + \beta a''(r^*)} < 0,$$

where the final inequality uses $a'(r^*) > 0$ and $-1 - \beta a''(r^*) < 0$. Writing $\bar{a}^* = a(r^*(\beta))$, we then have

$$\frac{\mathrm{d}\bar{a}^*}{\mathrm{d}\beta} = a'(r^*)\,\frac{\mathrm{d}r^*}{\mathrm{d}\beta} < 0.$$

□

Online Appendix

This appendix collects supplementary material: microfoundations for equilibrium selection and the payoff specification, robustness exercises and examples, and technical lemmas used to support some of the main-text proofs.

OA.1 Micro-founding the largest equilibrium

In Section 3 we focus on the largest equilibrium of the second stage (i.e. the one with the most participating agents). As discussed there, there are two natural interpretations of this focus, depending on the application. In entrenched-behavior settings, best-response dynamics from the all-participate profile select the largest equilibrium as a literal prediction. In coordination settings, the largest equilibrium characterizes the coordination capacity that the principal designs against.

Here, we formalize these two interpretations by providing two micro-foundations for the focus on the largest equilibrium. In the first, agents are boundedly rational and play myopic best responses. In the second, agents are fully rational and reach equilibrium using only local information and a short communication phase. Both micro-foundations yield the same outcome: the set of participating agents is the k-core of the realized graph.

Unsophisticated agents: myopic best-response dynamics

Set-up. Suppose that the game is played over $n+1$ periods. At $t = 1$, the principal chooses $r \geq 0$, and Nature realizes the graph G (as in our baseline model). Additionally, at $t = 1$, all agents participate *non-strategically* (i.e. $a_{i,t=1} = 1$ for all i). In each subsequent period $t \geq 2$, all agents best respond myopically to actions in the preceding period $t-1$. Agent i's action in period $t \geq 2$ is thus determined by the best response

$$BR^{(n)}_{i,t}(M_{i,t-1}) = \begin{cases} 1, & \text{if } M_{i,t-1} \geq k, \\ 0, & \text{if } M_{i,t-1} < k, \end{cases} \tag{18}$$

where $M_{i,t-1} = \sum_{j \neq i} G_{ij} a_{j,t-1}$. For convenience, we write $a_i \equiv a_{i,n+1}$ for i's action in the final period and assume that the principal cares about participation at the end of the game, not during it.

The result. Exactly as in our baseline model, the unique limit of the above dynamic process is that each agent participates if and only if he belongs to the k-core of the graph G. In other words, Remark 1 holds unchanged.

Discussion. In addition to myopia, this set-up assumes that all agents start off by participating. From a technical standpoint, this guarantees convergence of the best-response dynamics to the largest equilibrium of the static game. The logic is very similar to that laid out after Remark 1: agents iteratively drop out when they have fewer than k participating neighbors, and the process terminates at the k-core. The only difference is that agents are updating myopically rather than performing analogous reasoning in their heads.

Economically, this set-up imposes $a = 1$ as the pre-existing default. The myopic-updating view therefore best fits settings where the principal wants to eradicate or reduce pre-existing behaviors. This is natural when considering the adoption of new technologies or new social norms: the old versions were the status quo before the start of the game, so everyone must have been using the old technology or adhering to the old norm. Note that in this interpretation our terminology becomes less clean: "participate" means to use the *old* technology or norm, and "abstain" means to adopt the *new* one.

Sophisticated agents: local communication

At $t = 1$, the principal chooses $r \geq 0$, Nature realizes the graph G, and all agents announce $m_{i,1} \equiv \text{P}$. This is followed by a "communication phase" of n periods (periods 2 through $n+1$) and an "action phase" at $t = n+2$. In each period $t \geq 2$, agents observe only (i) how many neighbors they have and (ii) what each of those neighbors announced in the previous period.

Communication phase (periods 2 to n+1). During the communication phase, every agent i announces a *cheap talk* message $m_{i,t} \in \{P, A\}$, interpreted as "I Plan to participate" or "I will Abstain." Suppose agents adopt the rule:

$$m_{i,t} = \begin{cases} \text{A} & \text{if strictly fewer than } k \text{ neighbors sent P at } t-1, \\ \text{P} & \text{otherwise.} \end{cases}$$

Action phase (period n+2). Consider the strategy s_i given by: play $a_i = 1$ if and only if at least k of i's neighbors sent P in the final communication round (period $n+1$).

The result. The message rule implements the usual k-core peeling process: after (at most) n periods, no further agents switch to sending A's and the only agents who still announce P are the vertices in the k-core of G.

Given the strategy s_i for the action phase, any agent who still announces P in the final communication round has at least k neighbors who also announced P. These agents will therefore choose $a_i = 1$ in the action phase. Hence playing $a_i = 1$ is a best response for every agent who announces P at time $n+1$, that is, for every agent in the k-core. Any agent who observes fewer than k neighbors sending P in the final round plays $a_i = 0$, which is his best response since fewer than k of his neighbors will participate. Thus (while not strictly required by Nash equilibrium) $s = (s_i)_{i \in N}$ has the advantage of being sequentially rational in the action phase.

In the communication phase, no agent can profit (in any period) by unilaterally deviating from the message rule. Sending P when the rule prescribes A cannot push the number of remaining P-neighbors back to k, so it cannot raise the deviator's final payoff. Sending A when the rule prescribes P can only reduce the number of P-senders among the deviator's neighbors (by triggering further removals), weakly lowering his payoff. Therefore the prescribed message profile $m = (m_{i,t})_{i \in N, 1 \leq t \leq n+1}$ is itself incentive-compatible.

Consequently, (m, s) is a Nash equilibrium of the extensive-form game. The agents who play $a_i = 1$ in the terminal period are exactly those in the k-core of G.

Discussion. Agents here are fully rational and use the communication phase to perform the same iterated elimination that the "unsophisticated" agents achieve through myopic play. Crucially, each agent needs only *local* information: his own degree and the most recent messages of his neighbors. The procedure demonstrates that the k-core is a natural characterization of equilibrium even when agents are fully sophisticated, as in our baseline model (see Remark 1).

OA.2 Micro-founding the principal's benefit from the interaction rate

In Section 3 we assumed that the principal receives per-capita benefits of αr when she chooses an interaction rate r. Here, we open up that black box, and outline a game where agents choose bilateral actions with each of their friends.

The game. As in our baseline model, the principal chooses $r \geq 0$ at $t = 1$, and then Nature realizes the graph G (as described in Section 3). At $t = 2$, each agent simultaneously chooses whether or not to take a binary, and *neighbor-specific* action, $b_{ij} \in \{0, 1\}$ with each of his neighbors. This is in addition to the action $a_i \in \{0, 1\}$ present in the baseline model. Agents have preferences represented by a utility function $U_i = u_i + v_i$, where u_i is as in Equation (2) (from the baseline model) and v_i is:

$$v_i = \sum_{j \in \mathcal{N}_i} (\gamma b_{ij} + \delta b_{ij} b_{ji}), \tag{19}$$

where $\gamma > 0, \delta > 0$ and $\mathcal{N}_i := \{j : G_{ij}^{(n)} = 1\}$ is the set of i's neighbors. The principal then receives a benefit proportional to $\frac{1}{n} \sum_i \sum_j b_{ij}$—the per-capita number of neighbor-specific actions.

Discussion. Once agents have formed links (formally, after Nature has drawn the network), agents are choosing whether or not to "engage" with each of their neighbors in some way. For employees of a firm, this engagement may be conversations about work-relevant topics. For citizens, this may be conversations about job openings, product recommendations, or invitations to community events. For users of social media platforms, engagement may be interacting with a neighbor's post. But note that the precise form of engagement could differ across pairs of agents. For example, an employee of a firm could discuss different work-related topics with each of his neighbors.

We view these choices to engage as initiating relatively "low-stakes" interactions, and as ones where an agent can receive *some* benefit even if engagement is not reciprocated. Reciprocated interaction simply provides a higher benefit. Additionally, this engagement also provides some spillover benefits to the principal. For example, work-relevant conversations by employees help spread best practice within a firm—something the firm benefits from. Greater sharing of job openings or product recommendations by citizens can improve the functioning of labor and product markets—something that can benefit the government. And a social media platform sells more advertising when there is more engagement by users.

Outcomes. It is clear from the set-up that all agents i have a strictly dominant strategy for the neighbor-specific actions b_{ij}: choose $b_{ij} = 1$ for all $j \in \mathcal{N}_i$. So the principal's benefit $\frac{1}{n}\sum_i \sum_j b_{ij}$ is equal to $\frac{1}{n}\sum_i |\mathcal{N}_i|$, which is the average degree in the network ($|\mathcal{N}_i|$ is the number of neighbors i has—i.e., his degree). By a standard Law of Large Numbers, the average degree converges to the expected degree—which is equal to r by assumption. Each agent's expected degree is $r \times W_i$, and the W_i's are i.i.d. draws from a distribution F with $\mathbb{E}[W] = 1$. In short, we have $\lim_{n\to\infty} \frac{1}{n}\sum_i \sum_j b^*_{ij} = r$.

OA.3 Heavy-tail edge case (outside Assumption 1)

Lemma 2 is exactly where the finite-variance restriction is used to force $g(x) = x/\Phi_k(x) \to \infty$ as $x \downarrow 0$. If one leaves this moment class (for example, sufficiently heavy-tailed profiles with $\mathbb{E}[W^2] = \infty$), that boundary behavior can fail. Then the minimizer of g need not be interior, and participation can emerge continuously with $J_k(F) = 0$. We do not analyze that tail regime in this paper. All main-text results assume Assumption 1.

OA.4 Convex order does not imply monotonicity of the jump

The following example shows that ordering interaction profiles by the convex order does not guarantee monotonicity of the jump.

Remark 4 (Convex order counterexample for $k = 3$)**.** *Define two interaction profiles, each with* $\mathbb{E}[W] = 1$*:*

$$F_1: \quad W = \begin{cases} \frac{30}{49} & w.p.\ 0.7, \\ \frac{60}{49} & w.p.\ 0.1, \\ \frac{110}{49} & w.p.\ 0.2; \end{cases} \qquad F_2: \quad W = \begin{cases} \frac{10}{81} & w.p.\ 0.5, \\ \frac{130}{81} & w.p.\ 0.4, \\ \frac{240}{81} & w.p.\ 0.1. \end{cases}$$

One can verify that $F_1 \leq_{\text{cx}} F_2$ (i.e., F_2 is a mean-preserving spread of F_1) by checking the stop-loss order: $\mathbb{E}_{F_1}[(W-t)^+] \leq \mathbb{E}_{F_2}[(W-t)^+]$ for all $t \geq 0$.[30] *The variances are* $\mathrm{Var}(W) \approx 0.420$ *under F_1 and* $\mathrm{Var}(W) \approx 0.916$ *under F_2. However, direct computation gives*

$$J_3(F_1) \approx 0.1003 \quad < \quad J_3(F_2) \approx 0.1040.$$

The more dispersed profile has the larger *jump.*

The obstruction is that $J_k(F) = A_k(x_k^\star(F))$ depends on both the argmin $x_k^\star(F)$ of $g^F(x) = x/\Phi_k^F(x)$ and the composition with A_k^F. Neither the argmin map nor the composition preserves convex-order monotonicity.

OA.5 Robustness to a positive-degree periphery

This section shows that the comparative-statics results derived for the two-class family F_ν (Section 5) are robust to giving the periphery strictly positive interaction weights.

The softened family. For $\epsilon \in [0, 1)$, define

$$F_{\nu,\epsilon}: \quad W = \begin{cases} \epsilon & \text{w.p. } 1-\nu, \\ w_c(\nu,\epsilon) \equiv \dfrac{1-\epsilon(1-\nu)}{\nu} & \text{w.p. } \nu, \end{cases}$$

so that $\mathbb{E}[W] = 1$. Equivalently, $W = \epsilon + (1-\epsilon)X$ where $X \sim F_\nu$: every agent receives a uniform baseline weight ϵ, and the remaining mass is distributed according to the sharp two-class family. When $\epsilon = 0$, $F_{\nu,0} = F_\nu$. When $\epsilon > 0$, all agents have strictly positive expected degree (ϵr for the periphery, $w_c r$ for the core).

Key objects. Writing $\Phi_{k,\epsilon}(x) \equiv \mathbb{E}_{F_{\nu,\epsilon}}[W\psi_{k-1}(Wx)]$, $A_{k,\epsilon}(x) \equiv \mathbb{E}_{F_{\nu,\epsilon}}[\psi_k(Wx)]$, and $g_\epsilon(x) \equiv x/\Phi_{k,\epsilon}(x)$, the critical cutoff is $c_k(F_{\nu,\epsilon}) = \inf_{x>0} g_\epsilon(x)$ and the jump is $J_k(F_{\nu,\epsilon}) = A_{k,\epsilon}(x_\epsilon^\star)$, where $x_\epsilon^\star$ is the minimizer of g_ϵ.

[30] Both distributions have $\mathbb{E}[W] = 1$. Direct computation of $\mathbb{E}[(W-t)^+]$ for each distribution confirms the inequality for all $t \geq 0$, which is equivalent to the convex order.

Proposition 5 (Positive-degree periphery robustness)**.** *Fix $k \geq 3$ and $\nu \in (0,1)$. Assume $g_0(x) = x/\Phi_{k,0}(x)$ has a unique minimizer $x_0^\star$ (i.e., Assumption 2 holds for F_ν). For the rate claim in part (ii), assume in addition that $g_0''(x_0^\star) > 0$. Then:*

(i) (Continuity.) *$c_k(F_{\nu,\epsilon}) \to \nu\, c_k^{ER}$ and $J_k(F_{\nu,\epsilon}) \to \nu\, J_k^{ER}$ as $\epsilon \downarrow 0$.*

(ii) (Rates.) *The critical cutoff satisfies*

$$c_k(F_{\nu,\epsilon}) = \nu\, c_k^{ER} + O(\epsilon).$$

The jump is first-order stable:

$$J_k(F_{\nu,\epsilon}) = \nu\, J_k^{ER} + O(\epsilon),$$

where the implicit constants in $O(\epsilon)$ depend only on k and ν.

Proof. The argument proceeds in three steps.

Step 1: Uniform convergence on compacts. Since ψ_m is C^∞ and 1-Lipschitz, and the support of $F_{\nu,\epsilon}$ is bounded uniformly in ϵ (for fixed ν), $\Phi_{k,\epsilon}$ and $A_{k,\epsilon}$ converge uniformly to $\Phi_{k,0}$ and $A_{k,0}$ on every compact $[0,b]$:

$$\sup_{x\in[0,b]} \big|\Phi_{k,\epsilon}(x) - \Phi_{k,0}(x)\big| = O(\epsilon), \qquad \sup_{x\in[0,b]} \big|A_{k,\epsilon}(x) - A_{k,0}(x)\big| = O(\epsilon).$$

The bound follows from bounding the periphery term $(1-\nu)\epsilon\, \psi_{k-1}(\epsilon x) \leq (1-\nu)\epsilon$ and the shift in core weight $|w_c - 1/\nu| = O(\epsilon)$. The same holds for first derivatives.

Step 2: Minimizers stay bounded. For any fixed ν, $g_\epsilon(x) \to \infty$ as $x \downarrow 0$ (since $\Phi_{k,\epsilon}(x) = O(x^{k-1})$ for $k \geq 3$) and as $x \to \infty$ (since $\Phi_{k,\epsilon}(x) \to 1$), both uniformly in ϵ. Hence all minimizers of g_ϵ lie in a fixed compact $[a,b]$ for ϵ small.

Step 3: Argmin stability (parts (i) and (ii)). By Steps 1–2 and the uniqueness of $x_0^\star$, the Berge maximum theorem (applied to $-g_\epsilon$ on $[a,b]$) gives $x_\epsilon^\star \to x_0^\star$ and $c_k(F_{\nu,\epsilon}) = g_\epsilon(x_\epsilon^\star) \to g_0(x_0^\star) = \nu\, c_k^{ER}$. The Lipschitz bound from Step 1 gives the $O(\epsilon)$ rate for the cutoff.

For the jump, decompose

$$J_k(F_{\nu,\epsilon}) - \nu J_k^{ER} = \big[A_{k,\epsilon}(x_\epsilon^\star) - A_{k,0}(x_\epsilon^\star)\big] + \big[A_{k,0}(x_\epsilon^\star) - A_{k,0}(x_0^\star)\big].$$

The first bracket is $O(\epsilon)$ by Step 1 (uniform convergence of $A_{k,\epsilon}$ to $A_{k,0}$ on $[a,b]$). For the second bracket, use the nondegeneracy $g_0''(x_0^\star) > 0$. Since $g_0'(x_0^\star) = 0$, this yields a locally unique smooth critical-point branch under small C^1 perturbations of g_0'; combined with Step 1 (which gives $g_\epsilon' = g_0' + O(\epsilon)$ on $[a, b]$), we obtain $x_\epsilon^\star - x_0^\star = O(\epsilon)$. Because $A_{k,0}'$ is bounded on $[a, b]$, the second bracket is also $O(\epsilon)$. Hence

$$J_k(F_{\nu,\epsilon}) = \nu\, J_k^{ER} + O(\epsilon).$$

□

OA.6 The principal's optimal interaction rate is non-monotone in k

The discussion in Section 6 ("Raising the threshold") noted that the principal's optimal interaction rate r^* can move in either direction as the participation threshold k rises. The mechanism is a regime transition. For any (α, β, F) with $\alpha > c_k(F)$, the principal's optimum sits either at the cutoff itself or strictly above it (Proposition 3). When $\alpha \le c_k(F)$, the principal simply goes to the unconstrained naive optimum $r^{\text{naive}} = \alpha$: there is no need to consider participation. As k rises, the cutoff $c_k(F)$ rises; in the examples below, the jump $J_k(F)$ also rises. These changes can shift the principal between regimes, and the new r^* may lie above or below the previous one, depending on which transition is triggered. The two examples below illustrate. Both fix $\alpha = 5$ and $\beta = 1.8$.

Example 1: r^* rises with k. Take the homogeneous benchmark, $W \equiv 1$:

k	c_k	J_k	regime	r^*	$\bar{a}^*$
3	3.35	0.27	suppress at cutoff	3.35	0
4	5.15	0.44	naive	5.00	0
5	6.80	0.54	naive	5.00	0

At $k = 3$, the cutoff lies below the principal's naive optimum, and she pins r at the cutoff to preclude participation while keeping aggregate interaction as high as possible. As k rises to 4, c_k moves above α; the principal can reach the naive optimum without inducing any participation. So she does exactly that, and chooses $r^{\text{naive}} = \alpha = 5$. The transition raises r^* from 3.35 to 5.

Example 2: r^* falls with k. Take the Bernoulli dilution with $\nu = 0.3$: $W = 1/\nu$ with probability ν and $W = 0$ otherwise.

k	c_k	J_k	regime	r^*	$\bar{a}^*$
3	1.01	0.080	interior	5.00	0.30
8	3.40	0.206	interior	4.99	0.30
10	4.26	0.221	suppress at cutoff	4.26	0

For $k \leq 8$, the principal sits in the interior regime: she sets r just below the naive optimum α, accepts participation $\bar{a}^* \approx \nu = 0.3$ (the saturating share of the active core), and pays the participation cost $\beta \bar{a}_k^*$. As k rises to 10, the jump J_k has grown enough that the marginal cost of admitting *any* participation now exceeds the marginal benefit, and the principal flips to suppression at the new cutoff $c_{10} = 4.26$. The transition lowers r^* by roughly 0.7.[31]

Together, these examples confirm the claim made in Section 6: the comparative static of r^* in k depends on which regime transition the change triggers. A transition out of suppression-at-cutoff into the naive regime raises r^*, while a transition in the other direction lowers it.

OA.7 Call-price representation and proof of Theorem 4

We first state a general identity that underpins the comparison theorem.

Lemma 1 (Call-price identity)**.** *Let W be any nonnegative integrable random variable with distribution F, and define $C_F(t) \equiv \mathbb{E}[(W-t)_+]$ for $t \geq 0$. Let $h\colon [0,\infty) \to \mathbb{R}$ be C^2 with $h(0) = h'(0) = 0$ and $\int_0^\infty |h''(t)|\,dt < \infty$. Then*

$$\mathbb{E}[h(W)] = \int_0^\infty C_F(t)\, h''(t)\, dt.$$

Proof. For any fixed $w \geq 0$,

$$\int_0^\infty (w-t)_+ h''(t)\, dt = \int_0^w (w-t) h''(t)\, dt.$$

[31] Continuing this profile to higher k recovers Example 1's mechanism: at $k = 15$, $c_{15} \approx 6.30 > \alpha$, the cutoff is out of reach, and r^* rises back to $r^{\text{naive}} = \alpha = 5$. So within this single profile, r^* falls and then rises as k grows; the comparative static of r^* in k is non-monotone in both directions.

Integrating by parts,

$$\int_0^w (w-t)h''(t)\,dt = \Big[(w-t)h'(t)\Big]_0^w + \int_0^w h'(t)\,dt = -wh'(0) + h(w) - h(0).$$

Since $h(0) = h'(0) = 0$, this equals $h(w)$. Therefore

$$h(w) = \int_0^\infty (w-t)_+ h''(t)\,dt.$$

Now take expectations. Because $W \geq 0$ and $\mathbb{E}[W] < \infty$,

$$0 \leq C_F(t) = \mathbb{E}[(W-t)_+] \leq \mathbb{E}[W] < \infty.$$

Hence

$$\int_0^\infty C_F(t)\,|h''(t)|\,dt \leq \mathbb{E}[W] \int_0^\infty |h''(t)|\,dt < \infty,$$

so Fubini's theorem applies:

$$\mathbb{E}[h(W)] = \int_0^\infty \mathbb{E}[(W-t)_+]\,h''(t)\,dt = \int_0^\infty C_F(t)\,h''(t)\,dt. \qquad \square$$

Proof of Theorem 4. Write

$$p_m(z) \equiv e^{-z}\frac{z^m}{m!}, \qquad \psi_m(z) \equiv \mathbb{P}(\text{Poisson}(z) \geq m).$$

Recall the standard identities $\psi_m'(z) = p_{m-1}(z)$ and $p_m'(z) = p_{m-1}(z) - p_m(z)$.

Define, for fixed $x > 0$,

$$h_{k,x}(w) \equiv w\,\psi_{k-1}(xw), \qquad \ell_{k,x}(w) \equiv \psi_k(xw).$$

Because $k \geq 3$, we have $h_{k,x}(0) = h_{k,x}'(0) = 0$ and $\ell_{k,x}(0) = \ell_{k,x}'(0) = 0$. Also both second derivatives are polynomial-exponential functions and hence absolutely integrable on $[0,\infty)$.

For $h_{k,x}$, a direct calculation gives

$$h_{k,x}'(w) = \psi_{k-1}(xw) + xw\,p_{k-2}(xw),$$

hence, using $xw\, p_{k-3}(xw) = (k-2)p_{k-2}(xw)$,

$$h''_{k,x}(w) = \frac{x^{k-1}}{(k-2)!} w^{k-2} e^{-xw}(k - xw).$$

For $\ell_{k,x}$,

$$\ell'_{k,x}(w) = x\, p_{k-1}(xw),$$

so

$$\ell''_{k,x}(w) = \frac{x^k}{(k-1)!} w^{k-2} e^{-xw}(k - 1 - xw).$$

Applying Lemma 1 to $h_{k,x}$ and $\ell_{k,x}$ gives

$$\Phi_{k,F}(x) = \mathbb{E}[h_{k,x}(W)] = \frac{x^{k-1}}{(k-2)!} \int_0^\infty C_F(t)\, t^{k-2} e^{-xt}(k - xt)\, dt,$$
$$A_{k,F}(x) = \mathbb{E}[\ell_{k,x}(W)] = \frac{x^k}{(k-1)!} \int_0^\infty C_F(t)\, t^{k-2} e^{-xt}(k - 1 - xt)\, dt.$$

The one-sign-change claim is immediate: $t^{k-2}e^{-xt} > 0$ for all $t > 0$, while $k - xt$ crosses zero exactly once at $t = k/x$ and $k - 1 - xt$ crosses zero exactly once at $t = (k-1)/x$.

Subtracting the two formulas above and using $D(t) = C_{F_2}(t) - C_{F_1}(t)$ gives

$$\Phi_{k,F_2}(x) - \Phi_{k,F_1}(x) = \frac{x^{k-1}}{(k-2)!} \int_0^\infty D(t)\, t^{k-2} e^{-xt}(k - xt)\, dt. \tag{20}$$

Now assume (10). Evaluating (20) at $x = x_1^\star$ gives

$$\Phi_{k,F_2}(x_1^\star) \geq \Phi_{k,F_1}(x_1^\star),$$

hence

$$g_{F_2}(x_1^\star) \leq g_{F_1}(x_1^\star) = c_k(F_1).$$

Therefore

$$c_k(F_2) = \inf_{x>0} g_{F_2}(x) \leq g_{F_2}(x_1^\star) \leq c_k(F_1).$$

If moreover $D(t) = 0$ for all $t > k/x_1^\star$, then every nonzero contribution to the integral in (10) comes from the region where $k - x_1^\star t \geq 0$. Since also $D(t) \geq 0$, the integral is nonnegative. □

Proof of the call-price identity for Bernoulli dilution. Let $W^{(\nu)} = (B/\nu)W$ where $B \sim \text{Bernoulli}(\nu)$ is independent of W. Then

$$C_{F^{(\nu)}}(t) = \mathbb{E}\left[\left(\frac{B}{\nu}W - t\right)_+\right] = \nu\,\mathbb{E}\left[\left(\frac{W}{\nu} - t\right)_+\right] = \mathbb{E}[(W - \nu t)_+] = C_F(\nu t).$$

Substituting into the call-price representation for Φ_k and using the change of variables $u = \nu t$ gives $\Phi_{k,F^{(\nu)}}(x) = \Phi_{k,F}(x/\nu)$. Similarly, $A_{k,F^{(\nu)}}(x) = \nu\, A_{k,F}(x/\nu)$. Hence $g_{F^{(\nu)}}(x) = \nu\, g_F(x/\nu)$, so $c_k(F^{(\nu)}) = \nu\, c_k(F)$. If $x_k^\star(F)$ is the unique minimizer of g_F, then $\nu x_k^\star(F)$ is the unique minimizer of $g_{F^{(\nu)}}$, and

$$J_k(F^{(\nu)}) = A_{k,F^{(\nu)}}(\nu x_k^\star(F)) = \nu\, A_{k,F}(x_k^\star(F)) = \nu\, J_k(F). \qquad \square$$

OA.8 Supplementary proofs

This subsection collects the proofs of the technical lemmas that support—but are not particularly important in and of themselves—the proofs in Appendix A. They are placed online to keep the main manuscript within the page limit.

OA.8.1 Small-x behavior of Φ_k under finite variance

Lemma 2 (Small-x behavior of Φ_k under finite variance)**.** *Fix $k \geq 3$ and let $W \sim F$ satisfy $\mathbb{E}[W] = 1$, $\mathbb{P}(W > 0) > 0$, and $\mathbb{E}[W^2] < \infty$. Then*

$$\lim_{x \downarrow 0} \frac{\Phi_k(x)}{x} = 0.$$

Consequently, $g(x) = x/\Phi_k(x) \to \infty$ as $x \downarrow 0$, so any minimizer of g is interior.

Proof. Because $k \geq 3$, $\psi_{k-1}(y) \leq \psi_2(y)$ for all $y \geq 0$. Moreover, for $y \geq 0$, $\psi_2(y) = 1 - e^{-y}(1 + y) \leq \min\{1, y^2/2\}$. Hence

$$\Phi_k(x) = \mathbb{E}[W\psi_{k-1}(Wx)] \leq \mathbb{E}\big[W \min\{1, (Wx)^2/2\}\big]\,.$$

Split at $W \leq 1/x$:

$$\Phi_k(x) \leq \frac{x^2}{2}\mathbb{E}[W^3 \mathbf{1}_{\{W \leq 1/x\}}] + \mathbb{E}[W \mathbf{1}_{\{W > 1/x\}}].$$

For the tail term,

$$\mathbb{E}[W\mathbf{1}_{\{W>1/x\}}] \le x\,\mathbb{E}[W^2\mathbf{1}_{\{W>1/x\}}] = o(x),$$

since $\mathbb{E}[W^2] < \infty$ implies $\mathbb{E}[W^2\mathbf{1}_{\{W>t\}}] \to 0$ as $t \to \infty$.

For the truncated third moment, let $T = 1/x$. By integration by parts,

$$\mathbb{E}[W^3\mathbf{1}_{\{W\le T\}}] = 3\int_0^T t^2\,\mathbb{P}(W>t)\,dt - T^3\mathbb{P}(W>T) \;\le\; 3\int_0^T t^2\,\mathbb{P}(W>t)\,dt.$$

Since $\mathbb{E}[W^2] < \infty$, we have $t^2\mathbb{P}(W>t) \le \mathbb{E}[W^2\mathbf{1}_{\{W>t\}}] \to 0$. Therefore for any $\varepsilon > 0$ there exists T_0 such that $t^2\mathbb{P}(W>t) \le \varepsilon$ for all $t \ge T_0$, implying

$$\mathbb{E}[W^3\mathbf{1}_{\{W\le T\}}] \le 3\int_0^{T_0} t^2\mathbb{P}(W>t)\,dt + 3\varepsilon(T-T_0) = O(1) + o(T).$$

Thus $\mathbb{E}[W^3\mathbf{1}_{\{W\le T\}}] = o(T)$, so $(x^2/2)\mathbb{E}[W^3\mathbf{1}_{\{W\le 1/x\}}] = o(x)$. Combining terms gives $\Phi_k(x) = o(x)$, i.e. $\Phi_k(x)/x \to 0$, and hence $x/\Phi_k(x) \to \infty$. □

OA.8.2 Derivative identities for g and the H-criterion

Assumption 2 is a regularity condition on g: it requires g to have a unique minimum and to be strictly increasing past it. These conditions ensure that the giant k-core emerges through a single discontinuous jump as connectivity increases. For some interaction profiles satisfying Assumption 1, however, g can have multiple local minima—in such cases, the giant k-core emerges through several disjoint jumps, one as connectivity crosses each local minimum of g. The lemma below characterizes the critical points of g via the auxiliary function $H(x) \equiv x\,\Phi_k'(x) - \Phi_k(x)$. By part (ii), g has a unique critical point on $(0,\infty)$ if and only if H crosses zero exactly once, which gives a tractable single-crossing diagnostic for when Assumption 2 holds.

Lemma 3 (H-criterion for extrema of g)**.** *Fix $k \ge 3$ and an interaction profile F satisfying Assumption 1. Let*

$$\Phi_k(x) \equiv \mathbb{E}[W\,\psi_{k-1}(Wx)]\,, \qquad g(x) \equiv \frac{x}{\Phi_k(x)} \quad (x>0),$$

and define

$$H(x) \equiv x\,\Phi_k'(x) - \Phi_k(x).$$

Then:

(i) Φ_k *is twice continuously differentiable on* $(0, \infty)$*, with derivatives*

$$\Phi_k'(x) = \mathbb{E}\big[W^2\, p_{k-2}(Wx)\big]\,, \qquad \Phi_k''(x) = \mathbb{E}\big[W^3\,(p_{k-3}(Wx) - p_{k-2}(Wx))\big]\,,$$

where $p_m(z) \equiv \mathbb{P}(\text{Poisson}(z) = m)$*.*

(ii) g *is continuously differentiable on* $(0, \infty)$ *and*

$$g'(x) = \frac{\Phi_k(x) - x\Phi_k'(x)}{\Phi_k(x)^2} = -\frac{H(x)}{\Phi_k(x)^2}.$$

Consequently, $x > 0$ *is a critical point of* g *if and only if* $H(x) = 0$*.*

(iii) H *is continuously differentiable on* $(0, \infty)$ *and satisfies the identity*

$$H'(x) = x\,\Phi_k''(x).$$

Proof. Part (i): For each fixed $w \geq 0$,

$$\frac{\partial}{\partial x}\Big[w\,\psi_{k-1}(wx)\Big] = w^2\, p_{k-2}(wx).$$

Because $p_{k-2}(z) \leq 1$ for all $z \geq 0$ and $\mathbb{E}[W^2] < \infty$, dominated convergence yields

$$\Phi_k'(x) = \mathbb{E}\big[W^2\, p_{k-2}(Wx)\big]\,.$$

To differentiate once more, fix a compact interval $I = [a, b] \subset (0, \infty)$. For every integer $m \geq 0$ and every $z \geq 0$,

$$z\, p_m(z) = (m + 1)p_{m+1}(z) \leq m + 1.$$

Hence, for every $x \in I$,

$$W^3 p_m(Wx) = W^2\, \frac{Wx\, p_m(Wx)}{x} \leq \frac{m+1}{a}\, W^2.$$

Applying this with $m = k-3$ and $m = k-2$, we obtain

$$W^3 \big| p_{k-3}(Wx) - p_{k-2}(Wx) \big| \le \frac{2k-3}{a} W^2,$$

which is integrable by Assumption 1. Since

$$\frac{\partial}{\partial x}\Big[W^2 p_{k-2}(Wx)\Big] = W^3\big(p_{k-3}(Wx) - p_{k-2}(Wx)\big),$$

dominated convergence gives

$$\Phi_k''(x) = \mathbb{E}\big[W^3\big(p_{k-3}(Wx) - p_{k-2}(Wx)\big)\big], \qquad x \in I.$$

The same domination is uniform on I, so Φ_k'' is continuous on I. Since $I \subset (0, \infty)$ was arbitrary, $\Phi_k \in C^2((0, \infty))$.

Part (ii): This is the quotient rule:

$$g'(x) = \frac{\Phi_k(x) - x\Phi_k'(x)}{\Phi_k(x)^2} = -\frac{H(x)}{\Phi_k(x)^2}.$$

Hence $g'(x) = 0$ if and only if $H(x) = 0$.

Part (iii): By definition,

$$H(x) = x\Phi_k'(x) - \Phi_k(x),$$

so, using Part (i),

$$H'(x) = \Phi_k'(x) + x\Phi_k''(x) - \Phi_k'(x) = x\Phi_k''(x).$$

□

OA.8.3 Uniform small-x bound under bounded dispersion

Lemma 4 (Uniform small-x bound under bounded dispersion)**.** *Fix $k \ge 3$ and let $W \sim F$ satisfy $\mathbb{E}[W] = 1$ and $\mathrm{Var}(W) = \sigma^2 < \infty$. Then for all $x \in (0, 1/2]$,*

$$\Phi_k(x) = \mathbb{E}[W\psi_{k-1}(Wx)] \;\le\; 4x^2 + 6\sigma^2 x. \tag{21}$$

Consequently, for all $x \in (0, 1/2]$,

$$g(x) = \frac{x}{\Phi_k(x)} \;\geq\; \frac{1}{4x + 6\sigma^2}. \tag{22}$$

In particular, for any $a \in (0, 1/2]$,

$$\inf_{0<x\leq a} g(x) \;\geq\; \frac{1}{4a + 6\sigma^2}.$$

Proof. Because $k \geq 3$ we have $\psi_{k-1}(y) \leq \psi_2(y)$ for all $y \geq 0$. Moreover, if $N \sim \text{Poisson}(y)$ then $1_{\{N\geq 2\}} \leq N(N-1)/2$, so

$$\psi_2(y) = \mathbb{P}(N \geq 2) \leq \frac{\mathbb{E}[N(N-1)]}{2} = \frac{y^2}{2},$$

and hence $\psi_2(y) \leq \min\{1, y^2/2\}$. Therefore,

$$\Phi_k(x) \leq \mathbb{E}\left[W \min\left\{1, \frac{(Wx)^2}{2}\right\}\right] \leq \frac{x^2}{2}\mathbb{E}\big[W^3\mathbf{1}_{\{W\leq 1/x\}}\big] + \mathbb{E}\big[W\mathbf{1}_{\{W>1/x\}}\big].$$

Assume $x \leq 1/2$, so $1/x \geq 2$. For the first term, split at $W \leq 2$:

$$\mathbb{E}\big[W^3\mathbf{1}_{\{W\leq 1/x\}}\big] \leq \mathbb{E}\big[W^3\mathbf{1}_{\{W\leq 2\}}\big] + \mathbb{E}\big[W^3\mathbf{1}_{\{2<W\leq 1/x\}}\big] \leq 8 + \frac{1}{x}\mathbb{E}\big[W^2\mathbf{1}_{\{W>2\}}\big],$$

since $W^3 \leq (1/x)W^2$ on $\{W \leq 1/x\}$. On $\{W > 2\}$ we have $W - 1 \geq 1$, so Chebyshev implies $\mathbb{P}(W > 2) \leq \sigma^2$ and hence

$$\mathbb{E}\big[W^2\mathbf{1}_{\{W>2\}}\big] \leq 2\mathbb{E}[(W-1)^2] + 2\mathbb{P}(W > 2) \leq 4\sigma^2.$$

Thus

$$\frac{x^2}{2}\mathbb{E}\big[W^3\mathbf{1}_{\{W\leq 1/x\}}\big] \leq 4x^2 + 2\sigma^2 x.$$

For the second term, write $W = 1 + (W - 1)$ and use Cauchy–Schwarz and Chebyshev: for $t > 1$,

$$\mathbb{E}[W\mathbf{1}_{\{W>t\}}] \leq \mathbb{E}[(W-1)\mathbf{1}_{\{W>t\}}] + \mathbb{P}(W > t) \leq \sigma\sqrt{\mathbb{P}(W > t)} + \mathbb{P}(W > t) \leq \frac{\sigma^2}{t-1} + \frac{\sigma^2}{(t-1)^2}.$$

Taking $t = 1/x$ yields

$$\mathbb{E}\big[W \mathbf{1}_{\{W>1/x\}}\big] \le \frac{\sigma^2 x}{1-x} + \frac{\sigma^2 x^2}{(1-x)^2} \le 4\sigma^2 x,$$

where the last inequality uses $x \le 1/2$. Combining bounds gives (21). Dividing by x yields (22). □

OA.8.4 Uniform approximation on compacts

Lemma 5 (Uniform approximation on compacts)**.** *Fix $k \ge 3$ and $X > 0$, and let $W \sim F$ satisfy $\mathbb{E}[W] = 1$. Then*

$$\sup_{0\le x\le X} \big|\Phi_k(x) - \psi_{k-1}(x)\big| \le (1+X)\,\mathbb{E}|W-1|, \tag{23}$$

$$\sup_{0\le x\le X} \big|A_k(x) - \psi_k(x)\big| \le X\,\mathbb{E}|W-1|. \tag{24}$$

If additionally $\mathbb{E}[W^2] < \infty$, then

$$\sup_{0\le x\le X} \big|\Phi_k'(x) - \psi_{k-1}'(x)\big| \le \mathbb{E}|W^2-1| + X\,\mathbb{E}|W-1|. \tag{25}$$

Proof. We use only that ψ_m and p_m are 1-Lipschitz in their argument: since $\psi_m'(y) = p_{m-1}(y) \in [0,1]$ for $y \ge 0$, we have $|\psi_m(y) - \psi_m(z)| \le |y-z|$; similarly $p_m'(y) = p_{m-1}(y) - p_m(y) \in [-1,1]$ so $|p_m(y) - p_m(z)| \le |y-z|$.

Bound (23)*.* Write

$$\Phi_k(x) - \psi_{k-1}(x) = \mathbb{E}\big[W\psi_{k-1}(Wx) - \psi_{k-1}(x)\big].$$

Add and subtract $\psi_{k-1}(Wx)$:

$$W\psi_{k-1}(Wx) - \psi_{k-1}(x) = (W-1)\psi_{k-1}(Wx) + \big(\psi_{k-1}(Wx) - \psi_{k-1}(x)\big).$$

Since $\psi_{k-1}(\cdot) \in [0,1]$, the first term has absolute value at most $|W-1|$. By Lipschitzness, the second term is bounded by $|Wx - x| = x|W-1|$. Taking expectations yields $|\Phi_k(x) - \psi_{k-1}(x)| \le (1+x)\mathbb{E}|W-1|$, and taking $x \le X$ gives (23).

Bound (24). By Lipschitzness of ψ_k,

$$|A_k(x) - \psi_k(x)| = \big|\mathbb{E}[\psi_k(Wx) - \psi_k(x)]\big| \le \mathbb{E}[|Wx - x|] = x\,\mathbb{E}|W-1| \le X\,\mathbb{E}|W-1|.$$

Bound (25). Differentiate Φ_k:

$$\Phi_k'(x) = \mathbb{E}\big[W^2\,\psi_{k-1}'(Wx)\big] = \mathbb{E}\big[W^2\,p_{k-2}(Wx)\big], \qquad \psi_{k-1}'(x) = p_{k-2}(x).$$

Then

$$\Phi_k'(x) - p_{k-2}(x) = \mathbb{E}\big[(W^2-1)p_{k-2}(Wx)\big] + \mathbb{E}\big[p_{k-2}(Wx) - p_{k-2}(x)\big].$$

The first term is bounded by $\mathbb{E}|W^2-1|$ since $p_{k-2} \in [0,1]$. The second term is bounded by $\mathbb{E}|Wx - x| = x\mathbb{E}|W-1| \le X\mathbb{E}|W-1|$ by Lipschitzness of p_{k-2}. This yields (25). □

OA.8.5 Assumption 2 for sufficiently homogeneous F

Lemma 6. *Fix $k \ge 3$. There exists $\varepsilon_3 > 0$ such that any interaction profile F satisfying Assumption 1 and $\sigma(F) \le \varepsilon_3$ also satisfies Assumption 2, and the unique minimizer $x_k^\star(F)$ of $g^F(x) \equiv x/\Phi_k(x)$ satisfies $x_k^\star(F) \to x_k^{\star,ER}$ as $\sigma(F) \to 0$.*

Proof. Write $g^F(x) \equiv x/\Phi_k(x)$ and $g^{ER}(x) \equiv x/\psi_{k-1}(x)$, and let $x^\star \equiv x_k^{\star,ER}$ denote the unique minimizer of g^{ER} (existence and uniqueness are established in Step (a) below). Define

$$H^F(x) \equiv x\,\Phi_k'(x) - \Phi_k(x), \qquad H^{ER}(x) \equiv x\,\psi_{k-1}'(x) - \psi_{k-1}(x),$$

so that by Lemma 3(ii),

$$(g^F)'(x) = -\frac{H^F(x)}{\Phi_k(x)^2}, \qquad (g^{ER})'(x) = -\frac{H^{ER}(x)}{\psi_{k-1}(x)^2}.$$

The proof has four steps: (a) the strict sign pattern of H^{ER} on $(0,\infty)$, by direct calculation; (b) transfer of this sign pattern to H^F on a compact interval $[a,b]$, giving at least one zero of H^F near $x^\star$; (c) strict monotonicity of H^F on $[x^\star - \delta, b]$, upgrading existence to uniqueness; and (d) extension of strict increase of g^F from $[x_k^\star(F), b]$ to all of $[x_k^\star(F), \infty)$.

Step (a): sign pattern of H^{ER} (by direct calculation). Since $\psi'_{k-1}(x) = p_{k-2}(x)$, we have $H^{ER}(x) = x\, p_{k-2}(x) - \psi_{k-1}(x)$. Differentiating and using $p'_m(x) = p_{m-1}(x) - p_m(x)$ gives $(H^{ER})'(x) = p_{k-2}(x) + x\, p'_{k-2}(x) - \psi'_{k-1}(x) = x(p_{k-3}(x) - p_{k-2}(x))$. Moreover $p_{k-3}(x)/p_{k-2}(x) = (k-2)/x$, so $p_{k-3}(x) - p_{k-2}(x)$ is strictly positive for $x < k-2$ and strictly negative for $x > k-2$. Hence H^{ER} is strictly increasing on $(0, k-2]$ and strictly decreasing on $[k-2, \infty)$. Since $H^{ER}(x) \to 0$ as $x \downarrow 0$ and H^{ER} is strictly increasing on $(0, k-2]$, we have $H^{ER}(k-2) > 0$; and $H^{ER}(x) \to -1$ as $x \to \infty$ because $p_{k-2}(x) \to 0$ and $\psi_{k-1}(x) \to 1$. Therefore H^{ER} has a unique zero on $(0, \infty)$, necessarily at some $x^\star > k-2$. In particular, $H^{ER}(x) > 0$ for $x \in (0, x^\star)$ and $H^{ER}(x) < 0$ for $x > x^\star$, so g^{ER} is strictly decreasing on $(0, x^\star)$ and strictly increasing on $(x^\star, \infty)$.

Step (b): transfer the single-crossing of H^{ER} to H^F on a compact interval. By Step (a), $x^\star > k-2$. Choose $\delta > 0$ with $x^\star - \delta > k-2$, a with $0 < a < x^\star - \delta$ (e.g., $a = (k-2)/2$), and b with $b > x^\star + \delta$ large enough that $\psi_{k-1}(b/2) \geq 1/2$ and $k^k e^{-k}/((k-2)!\, b) \leq 1/8$ (we use these bounds in Step (d)). By continuity and the strict sign pattern in Step (a),

$$m_L \equiv \min_{x \in [a, x^\star - \delta]} H^{ER}(x) > 0, \qquad m_R \equiv \min_{x \in [x^\star + \delta, b]} \big(- H^{ER}(x) \big) > 0.$$

Next note the deterministic bound, for all $x \in [a, b]$,

$$|H^F(x) - H^{ER}(x)| \leq x\, |\Phi'_k(x) - \psi'_{k-1}(x)| + |\Phi_k(x) - \psi_{k-1}(x)| \leq b \sup_{[0,b]} |\Phi'_k - \psi'_{k-1}| + \sup_{[0,b]} |\Phi_k - \psi_{k-1}|.$$

By Lemma 5 (and $\mathbb{E}|W - 1| \leq \sigma(F)$, $\mathbb{E}|W^2 - 1| \leq \sigma(F)\sqrt{4 + \sigma(F)^2}$), the right-hand side tends to 0 as $\sigma(F) \to 0$. Hence, for $\sigma(F)$ sufficiently small,

$$H^F(x) \geq \frac{m_L}{2} \text{ for all } x \in [a, x^\star - \delta], \qquad H^F(x) \leq -\frac{m_R}{2} \text{ for all } x \in [x^\star + \delta, b]. \quad (26)$$

In particular $H^F(x^\star - \delta) > 0$ and $H^F(x^\star + \delta) < 0$, so by continuity H^F has at least one zero in $(x^\star - \delta, x^\star + \delta)$.

Step (c): uniqueness via monotonicity of H^F on $[x^\star - \delta, b]$. For $x > 0$, differentiating $\Phi'_k(x) = \mathbb{E}[W^2 p_{k-2}(Wx)]$ under the expectation (as in Lemma 3(i)) gives $\Phi''_k(x) = \mathbb{E}[W^3(p_{k-3}(Wx) - p_{k-2}(Wx))]$. Let $q(z) \equiv p_{k-3}(z) - p_{k-2}(z) = \psi''_{k-1}(z)$. Because $x^\star - \delta > k-2$, q is continuous and strictly negative on $[x^\star - \delta, b]$, so

$m'' \equiv \min_{x\in[x^\star-\delta,b]}(-q(x)) > 0$. For each $x \in [x^\star - \delta, b]$, define $f_x(w) \equiv w^3 q(wx)$. Using the explicit Poisson form $p_m(t) = e^{-t}t^m/m!$, every term in $f'_x(w)$ is a polynomial in w times e^{-wx}; since $x \geq x^\star - \delta > 0$, this implies

$$L \equiv \sup_{x\in[x^\star-\delta,b]} \sup_{w\geq 0} |f'_x(w)| < \infty,$$

because $\sup_{w\geq 0} w^m e^{-w(x^\star-\delta)} < \infty$ for each fixed m. Therefore, for all $x \in [x^\star - \delta, b]$,

$$|\Phi''_k(x) - q(x)| = |\mathbb{E}[f_x(W) - f_x(1)]| \leq L\,\mathbb{E}|W-1| \leq L\,\sigma(F).$$

Choosing $\sigma(F)$ small enough that $L\sigma(F) \leq m''/2$ yields $\Phi''_k(x) \leq -m''/2 < 0$ on $[x^\star - \delta, b]$. Consequently, by Lemma 3(iii), $(H^F)'(x) = x\,\Phi''_k(x) < 0$ for all $x \in [x^\star - \delta, b]$, so H^F is strictly decreasing there. Combined with (26), this implies that H^F has a *unique* zero in $(x^\star - \delta, x^\star + \delta)$; denote it by $x^\star_k(F)$. Since $(g^F)'(x) = -H^F(x)/\Phi_k(x)^2$, we have $(g^F)'(x) < 0$ for $x < x^\star_k(F)$ and $(g^F)'(x) > 0$ for $x \in (x^\star_k(F), b]$, hence g^F is strictly decreasing on $[a, x^\star_k(F)]$ and strictly increasing on $[x^\star_k(F), b]$. Moreover, since $\delta > 0$ was arbitrary, this shows $x^\star_k(F) \to x^\star$ as $\sigma(F) \to 0$.

Step (d): strict increase on $[x^\star_k(F), \infty)$. Step (c) gives strict increase on $[x^\star_k(F), b]$. We extend this to $[b, \infty)$ using the conditions $\psi_{k-1}(b/2) \geq 1/2$ and $k^k e^{-k}/((k-2)!\,b) \leq 1/8$ chosen in Step (b). Then for any F satisfying Assumption 1 and any $x \geq b$,

$$\Phi_k(x) \geq \mathbb{E}\big[W\,\psi_{k-1}(Wx)\mathbf{1}_{\{W\geq 1/2\}}\big] \geq \psi_{k-1}(b/2)\,\mathbb{E}\big[W\mathbf{1}_{\{W\geq 1/2\}}\big] \geq \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4},$$

where we used $Wx \geq x/2 \geq b/2$ on $\{W \geq 1/2\}$ and $\mathbb{E}[W\mathbf{1}_{\{W<1/2\}}] \leq 1/2$. Also, using $p_{k-2}(t) = e^{-t}t^{k-2}/(k-2)!$ and $\sup_{w\geq 0} w^k e^{-wx} = (k/x)^k e^{-k}$, let $C_k \equiv k^k e^{-k}/(k-2)!$. Then

$$x\Phi'_k(x) = x\,\mathbb{E}[W^2 p_{k-2}(Wx)] = \frac{x^{k-1}}{(k-2)!}\,\mathbb{E}\big[W^k e^{-Wx}\big] \leq \frac{x^{k-1}}{(k-2)!}\cdot\frac{k^k e^{-k}}{x^k} = \frac{C_k}{x} \leq \frac{C_k}{b} \leq \frac{1}{8}.$$

Thus $\Phi_k(x) - x\Phi'_k(x) \geq 1/8 > 0$ for all $x \geq b$, so $(g^F)'(x) > 0$ on $[b, \infty)$. Therefore g^F is strictly increasing on $[x^\star_k(F), \infty)$.

By Steps (a)–(d), g^F has the unique minimizer $x^\star_k(F)$ and is strictly increasing on $[x^\star_k(F), \infty)$, so Assumption 2 holds. The convergence $x^\star_k(F) \to x^\star$ was established in Step (c).

□

OA.8.6 Existence of the threshold $\bar{\beta}$

Lemma 7. *Suppose $\alpha > c_k(F)$. There exists a unique $\bar{\beta}(\alpha; F) \in (0, \infty)$ such that the function f defined in (12) satisfies*

$$f(\bar{\beta}) = u(c_k(F)), \qquad f(\beta) > u(c_k(F)) \text{ for } \beta < \bar{\beta}, \qquad f(\beta) < u(c_k(F)) \text{ for } \beta > \bar{\beta},$$

where $u(r) \equiv \alpha r - \frac{1}{2}r^2$.

Proof. Write $c \equiv c_k(F)$ and $J \equiv J_k(F) > 0$. By Theorem 2(i) and the definition of J, $a(r) \geq J$ for all $r > c$. Hence for any $0 \leq \beta_1 < \beta_2$ and any $r > c$, $\pi_{\beta_1}(r; F) - \pi_{\beta_2}(r; F) = (\beta_2 - \beta_1)a(r) \geq (\beta_2 - \beta_1)J$. Taking suprema gives $f(\beta_1) \geq f(\beta_2) + (\beta_2 - \beta_1)J > f(\beta_2)$, so f is strictly decreasing. It is also finite and convex (as the pointwise supremum of affine functions of β), and hence continuous on $(0, \infty)$.

Recall $r^{\text{naive}} = \alpha > c$ is the unconstrained maximizer of u. Then $f(0) = \sup_{r>c} u(r) = u(r^{\text{naive}}) > u(c)$, while $f(\beta) \leq u(r^{\text{naive}}) - \beta J \to -\infty$ as $\beta \to \infty$. By the intermediate value theorem and strict monotonicity, there exists a unique $\bar{\beta} \in (0, \infty)$ with the stated properties. □

OA.8.7 Smoothness and strict monotonicity of participation above the cutoff

Lemma 8 (Smoothness of participation above the cutoff)**.** *Under Assumption 2 the following holds: For each $r > c \equiv c_k(F)$, let $x(r)$ be the unique solution to $x = r\Phi_k(x)$ in $[x_k^\star(F), \infty)$, and recall $a(r) \equiv a_k(r; F) = A_k(x(r))$.*

Then $r \mapsto x(r)$ and $r \mapsto a(r)$ are C^2 on (c, ∞). Moreover, for all $r > c$,

$$a'(r) = A_k'(x(r))\, x'(r) > 0.$$

In particular, $a(\cdot)$ is strictly increasing on (c, ∞), and for each $\beta > 0$ the map $r \mapsto \pi_\beta(r; F) = \alpha r - \frac{1}{2}r^2 - \beta\, a(r)$ is C^2 on (c, ∞).

Proof. Write $p_m(z) \equiv \mathbb{P}(\text{Poisson}(z) = m) = e^{-z} z^m / m!$.

Φ_k is C^2 on $(0, \infty)$. By Lemma 3(i), $\Phi_k \in C^2((0, \infty))$ with derivatives $\Phi_k'(x) = \mathbb{E}[W^2\, p_{k-2}(Wx)]$ and $\Phi_k''(x) = \mathbb{E}[W^3(p_{k-3}(Wx) - p_{k-2}(Wx))]$.

A_k is C^2 on $(0, \infty)$. Since $\psi_k'(z) = p_{k-1}(z)$ and $|p_{k-1}(z)| \leq 1$, dominated convergence (with integrable dominator W) gives, for every $x > 0$, $A_k'(x) = \mathbb{E}[W\, p_{k-1}(Wx)]$.

For the second derivative, $\partial_x[W\,p_{k-1}(Wx)] = W^2(p_{k-2}(Wx) - p_{k-1}(Wx))$, and $W^2|p_{k-2}(Wx) - p_{k-1}(Wx)| \le 2W^2$, which is integrable by $\mathbb{E}[W^2] < \infty$ (Assumption 1). Dominated convergence therefore yields $A_k''(x) = \mathbb{E}[W^2(p_{k-2}(Wx) - p_{k-1}(Wx))]$, and A_k'' is continuous by the same domination, so $A_k \in C^2((0,\infty))$.

For $r > c$, Assumption 2 implies that the equation $r = g(x)$ has a unique solution $x = x(r) \in [x_k^\star(F), \infty)$, and this $x(r)$ is exactly the largest-fixed-point branch solving $x = r\Phi_k(x)$. Since $c = g(x_k^\star(F))$ and g is strictly increasing on $[x_k^\star(F), \infty)$, we have $x(r) > x_k^\star(F)$ for all $r > c$. By Assumption 2, $g'(x(r)) > 0$. Now $g'(x) = (\Phi_k(x) - x\Phi_k'(x))/\Phi_k(x)^2$, so $g'(x(r)) > 0$ implies $\Phi_k(x(r)) > x(r)\Phi_k'(x(r))$. Using $x(r) = r\Phi_k(x(r))$, this is equivalent to $1 - r\Phi_k'(x(r)) > 0$. Hence the implicit function theorem applied to $H(r,x) \equiv x - r\Phi_k(x) = 0$ gives that $r \mapsto x(r)$ is C^2 on (c,∞), with derivative $x'(r) = \Phi_k(x(r))/(1 - r\Phi_k'(x(r))) > 0$. Since A_k is C^2, $a(r) = A_k(x(r))$ is C^2 on (c,∞), and $a'(r) = A_k'(x(r))\,x'(r)$. Finally, $A_k'(x(r)) > 0$ because $\mathbb{P}(W > 0) > 0$ (Assumption 1) and $p_{k-1}(Wx(r)) > 0$ whenever $W > 0$ and $x(r) > 0$. Thus $a'(r) > 0$ for all $r > c$. □

OA.8.8 Generic uniqueness of the principal's optimum

Lemma 9 (Generic uniqueness)**.** *Fix $\alpha > 0$ and define the value function*

$$V(\beta) \equiv \sup_{r \ge 0} \Big(\alpha r - \tfrac{1}{2}r^2 - \beta\, a_k(r;F)\Big).$$

Then V is convex in β, hence differentiable for all but countably many β. Moreover, if V is differentiable at β, then $\arg\max_{r\ge 0} \pi_\beta(r;F)$ is a singleton.

Proof. For each fixed r, $\beta \mapsto \pi_\beta(r;F)$ is affine with slope $-a_k(r;F)$, so $V(\beta) = \sup_r \pi_\beta(r;F)$ is convex as the pointwise supremum of affine functions. A convex function on an interval is differentiable except at a countable set of points.

Now fix β and suppose there are two distinct global maximizers $r_1 < r_2$. If $r_2 \le c_k(F)$, then $a_k(r_1;F) = a_k(r_2;F) = 0$ and $\pi_\beta(r;F) = \alpha r - \frac{1}{2}r^2$ on $[0, c_k(F)]$, which has a unique maximizer on $[0, c_k(F)]$ (equal to $\min\{\alpha, c_k(F)\}$), a contradiction. Hence $r_2 > c_k(F)$, and by strict monotonicity of $a_k(\cdot;F)$ on $(c_k(F), \infty)$ (Theorem 2(i)), we have $a_k(r_1;F) \ne a_k(r_2;F)$. Therefore the two affine functions $\beta \mapsto \pi_\beta(r_1;F)$ and $\beta \mapsto \pi_\beta(r_2;F)$ have different slopes, so V cannot be differentiable at β. □