Controlling Complex Contagions*

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Abstract

Many social and economic behaviors, from technology adoption to protest participation, are complex contagions requiring social reinforcement. In these settings, agents take an action only if a minimum number of their neighbors also do so. While threshold models capture this core mechanism, deriving tractable comparative statics for how behavior responds to network structure has proven elusive, rendering policy recommendations for network design intractable. We introduce a new approach using large random networks. This unlocks powerful comparative statics and allows us to solve for the optimal network design by a principal who influences overall network connectivity but not its exact structure. We find that as connectivity increases, participation jumps discontinuously from zero at a critical cut-off and exhibits diminishing returns thereafter. This reveals a "missing middle" in the principal's optimal choice: network connectivity is either set at the critical threshold to prevent participation, or pushed substantially above it.

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1 Introduction

In many economic settings, an individual will take an action only when enough of their peers do. An employee adopts a new workplace norm only if enough of his closest colleagues do; a citizen joins a protest only when a critical mass of friends will provide "safety in numbers"; an individual continues speaking a minority language only when enough peers speak it too. These phenomena, known as *complex contagions*, are central to understanding behaviors ranging from corporate culture and technology adoption, to political mobilization and the spread of social norms.[1]

An important feature emerging from empirical work is that behavior in these settings can sometimes change suddenly in response to modest changes in the environment. For example, controlled network experiments show that a committed minority of roughly 25% can flip a prevailing social convention; just below this cutoff, change rarely occurs, while just above it, adoption spreads rapidly (Centola et al., 2018). Outside the lab, bank runs can propagate along interbank exposures, with deposit withdrawals rising steeply at institutions more exposed to a failing bank (Iyer and Peydró, 2011). This "tipping point" dynamic (Schelling, 1978)—where modest changes in connectivity or early participation produce outsized changes in outcomes—has also been observed in political mobilization (Madestam et al., 2013; Enikolopov et al., 2020).

A more limited empirical literature investigates the network *design* choices of principals—such as governments or firms—who have some control over interactions among agents. Such principals appear to adopt "extreme" strategies. For example, King et al. (2013) find that Chinese government censorship focuses on suppressing coordination and the spread of actions, but still tolerates broad criticism. This is suggestive of a policy targeted at keeping activity below a tipping point. More starkly, the growing use of internet shutdowns during protests or contested elections represents an extreme choice to dramatically reduce connectivity, forgoing its economic benefits in order to eliminate the possibility of digitally coordinated collective action (Access Now, 2022; V-Dem Institute, 2024).

[1] A substantial body of empirical work has documented the importance of social reinforcement in settings as varied as the adoption of new agricultural technologies (Bandiera and Rasul, 2006; Beaman et al., 2021), communication technologies (Björkegren, 2019), birth control use (Munshi and Myaux, 2006), health behaviors (Christakis and Fowler, 2007, 2008), and protest participation (Larson et al., 2019; González, 2020; Bursztyn et al., 2021).

Despite this accumulating evidence, existing theories do not give us a clean map from network structure to behavior. Fixed-network analyses deliver rich microfoundations but make it hard to obtain clean comparative statics with respect to network structure, while more tractable approaches to date push the network into the background (either by using mean-field approximations or abstracting from the network altogether). As a result, in complex-contagion environments we still lack a simple benchmark that links explicit network structure to equilibrium behavior and to a principal's optimal design. Our contribution is to provide such a benchmark and to make the relevant comparative statics and design problem tractable.

We develop a model that is consistent with these empirical regularities and that clarifies the connection between a sharp tipping point in participation and extreme strategies in optimal design. In our framework, the discontinuous onset of participation (which arises from a network property with a sharp threshold) means that intermediate levels of connectivity are never optimal for a principal.[2] Optimal policies place either exactly at the cutoff or well above it. This is because the principal in our setting dislikes participation but benefits from network connectivity. So the discontinuous jump up in participation must be made up for by even more network connectivity.

More concretely, we consider a setting where the principal can influence the overall connectivity of a network formed among a large number of agents, but cannot control its exact structure. This captures a realistic constraint: a principal can influence the environments for interaction (e.g., through communication platforms, public forums, or open-plan offices) but cannot micromanage individual relationships. Focusing on the limit of a large random network allows us to fully characterize equilibrium behavior and solve the principal's design problem, yielding precise and testable predictions.

Our first set of results characterizes the relationship between network connectivity and equilibrium participation—the fraction of agents who take the action. Below a critical cutoff, participation is zero. Just above this cutoff, participation jumps discontinuously—to at least a quarter of all agents. We demonstrate that both the critical connectivity cutoff and the size of this discontinuous

[2] This mechanism is distinct from other sources of discontinuity studied elsewhere (e.g., those arising from input complementarities in supply networks); we discuss those connections in the Related Literature and emphasize that here the discontinuity is a primitive of complex contagion itself.

jump are increasing in the number of neighbors each agent needs to take the action for them to want to do so too (which we call the participation threshold, k). Beyond the jump, participation is a strictly concave function of connectivity. These sharp theoretical predictions are derived by analyzing the properties of the network's giant k-core—the sub-graph of agents whose participation is endogenously sustained by mutual peer support.[3] The giant k-core has been studied in pure mathematics, but our results on its monotonicity and concavity are new, and are the key drivers of the economic outcomes.

Our second contribution is to solve the principal's optimal choice of network connectivity. This reveals a "missing middle" in her strategy. Suppose the principal receives some benefits from connectivity but finds participation by agents costly. The discontinuous jump in participation forces a stark trade-off: to accept any participation is to accept a large amount of it. Consequently, the principal's optimal choice is generically unique and located at one of two extremes. She either sets connectivity precisely at the critical threshold, ensuring zero participation. Or, she chooses a substantially higher level of connectivity where the intrinsic benefits are large enough to outweigh the costs of widespread participation. Intermediate levels of connectivity are never optimal; the principal is "in for a penny, in for a pound."

Finally, our analysis shows why complex contagions are so resistant to random seeding—a common policy intervention. We show that random seeding has a direct, linear, effect on the agents who are seeded—but fails to trigger a cascade among other agents. Our model precisely quantifies this failure: unless the seeding is large enough to eliminate a giant k-core on its own, it has almost no impact. Our findings complement the work of Jackson and Storms (forthcoming) and formalizes the empirical intuition that interventions must be targeted and create local density, not just isolated adopters, to succeed.

A Roadmap. The rest of the paper is organized as follows. Section 2 discusses related literature. To build intuition Section 3 presents a simple example using a fixed network. Section 4 sets out the model. Section 5 characterizes agents' behavior. Section 6 characterizes the principal's behavior.

[3] The 'giant k-core' is the large random networks analogue of the 'k-core'. The k-core is known to be critical in characterizing equilibrium behavior in threshold games on finite, deterministic networks (Gagnon and Goyal, 2017; Langtry et al., 2024).

Section 7 examines other interventions the principal can use. Section 8 discusses some directions for future work and concludes. All proofs are deferred to Appendix A.

2 Related Literature

Our paper makes two main contributions. First, we use random networks to study complex contagions—settings where an agent's action depends on the actions of several peers—and to derive sharp, tractable comparative statics for equilibrium behavior. These comparative statics have proven elusive in traditional fixed-network models. Second, we embed this analysis within a principal-agent framework to solve for the optimal design of a network when the principal has control over its connectivity, but not over the specific microstructure of links. This delivers sharp predictions and policy insights for a mechanism central to many economic settings. Our work builds on and contributes to three main areas of literature.

First, there is a large literature that studies models of complex contagions (sometimes called threshold models), with applications to a very wide range of economic behaviors.[4] These models primarily use a fixed network (Gagnon and Goyal, 2017; Reich, 2023), and often focus on the diffusion of the action from some starting group of agents (Morris, 2000; Acemoglu et al., 2011).[5] Our contribution is to provide precise comparative statics and network design insights that have proved elusive in the fixed network setup. On a more technical front, we also provide new mathematical results about the concavity and monotonicity of the *giant k-core*.

An important exception in this literature is Jackson and Storms (forthcoming), who use stochastic block models in a complex contagion environment. Their focus is very different to ours. Their goal is to understand (a) which groups of agents must take the same action in any equilibrium, and (b) how to use this knowledge to best pick a set of initial adopters to maximize the spread

[4]These include technology adoption (Reich, 2023), protest (Chwe, 2000), pricing (Zhang, 2025), financial contagion (Rogers and Veraart, 2013; Elliott et al., 2014), persuasion (Candogan, 2022), and choices of whether to participate in formal markets Gagnon and Goyal (2017). Granovetter (1978) and Schelling (1978) also suggest many other applications.

[5]Another approach has been to push the network into the background—by abstracting from it altogether (Granovetter, 1978; Schelling, 1978) or by imposing a 'mean field' assumption (Jackson and Yariv, 2006; López-Pintado, 2006, 2008)—or to assume that agents only have partial information about the game (Galeotti et al., 2010; Leister et al., 2022). In contrast to these approaches, we keep the network very much in the foreground and maintain complete information.

of a new behavior. Our focus is instead on comparative statics for overall equilibrium behavior with respect to features of the network, as well as how a principal would design the network. We discuss the implications of Jackson and Storms (forthcoming) with respect to seeding in our model in Section 7.

Second, there is a small but growing literature that uses random networks to model social and economic behavior. Most closely related to us is the strand of work that considers 'simple contagion' settings—where agents only need *one* of their neighbors to take an action for them to be willing to do so too. These models are typically used to explain behaviors driven by the spread of information (Campbell, 2013; Sadler, 2020; Akbarpour et al., 2020; Campbell et al., 2024a,b; Langtry, 2025). Simple contagions are known to behave very differently from complex contagions (Centola and Macy, 2007; Centola, 2010), and existing models are therefore ill-suited for the behaviors we focus on. We develop a tractable model of complex contagions on random networks with absolute (k-neighbor) thresholds.

Within the literature on random networks, our work also contributes to a technical literature on 'discontinuous phase transitions'—settings where small changes in network structure around some critical cut-off induce a discontinuous change in equilibrium behavior. Watts (2002) analyzed seed-driven cascades with fractional thresholds on random graphs. There, a discontinuous phase transition arises when the network becomes dense enough that the "vulnerable cluster" loses percolation. Work by Buldyrev et al. (2010) has shown that these transitions can be driven by interactions between interdependent networks. Separately, Elliott et al. (2022) show how such transitions can arise from a need for multiple types of input in a supply network, doing so in a setting with endogenous network formation. We identify an alternative mechanism: discontinuous phase transitions arise directly from the threshold-based nature of complex contagion itself.

Third, we contribute to the literature on network design and formation. This literature has increasingly focused on settings where a network forms and then agents play a game on the resulting network Galeotti and Goyal (2010); Sadler and Golub (2021); Kinateder and Merlino (2017, 2022). In part due to technical challenges, there is little work on network formation in a random networks setting. Elliott et al. (2022) and Langtry (2025) are exceptions to this. However, these papers consider endogenous formation by the agents. In contrast, we consider optimal design by a principal.

Before presenting the formal model and analysis, we begin with a simple, illustrative example using a small number of agents and a principal who can control the exact network structure. To help fix ideas, we focus on the specific example of a government seeking to quash political protest among its citizens. This example captures the main trade-offs the principal (government) faces, and showcases many results which appear intuitive. However, it also demonstrates a critical limitation of using fixed networks: these intuitive results cannot be established rigorously because comparative statics are weak. We can say very little about how equilibrium behavior responds in general to changes in the network structure, or to the principal's incentives.

3 An Illustrative Example

Set-up. Citizens $N = \{1, \ldots, \mathbf{8}\}$ interact in a social network. A government oversees these citizens, and can control the links between them in the network. Links between citizens raise productivity—from which the government benefits—but also help coordinate protest. The productivity benefits are concave in the number of links. Each citizen prefers to protest if and only if at least **3** of her neighbors in the network do so too.[6] Assume that, given the network, citizens are able to coordinate on the largest equilibrium protest. The cost of protest to the government is proportional to the fraction of citizens who protest. Letting L be the number of links, and $\bar{a}$ be the fraction of citizens who protest, the government's payoff is:

$$\pi(L) = \underbrace{\alpha L - L^2}_{\text{productivity benefits}} - \underbrace{\beta\,\bar{a}}_{\text{protest costs}}, \quad \text{with } \alpha, \beta > 0. \tag{1}$$

For the purposes of this example, we take $\alpha = 36$.

Who protests? Any citizen with fewer than 3 neighbors will never protest. Suppose we remove these citizens from the network and consider who will protest in the reduced network. By the same logic, any citizen with fewer than 3 neighbors in the reduced network will never protest. If

[6] Games where an agent takes an action if and only if a certain number, or fraction, of others do so too are often called 'threshold games'.

we repeat this process until a further iteration does not remove any citizens, those who 'survive' are precisely the citizens who will protest. This process finds the citizens in the network's *3-core*: the largest induced sub-network in which every citizen has degree at least 3. Thus, in the largest equilibrium of this threshold game, citizens protest if and only if they belong to the 3-core.

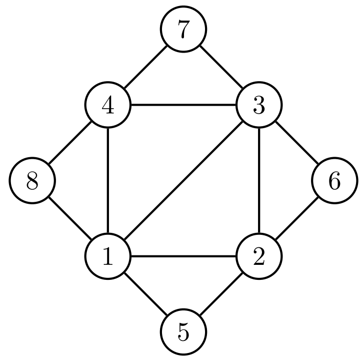
How many? With 8 citizens, the maximum number of links possible *without* a 3-core is 13.[7] We show such a network in Figure 1A. Adding any 14^{th} link must create a 3-core, as shown in Figure 1B. Because every citizen in the 3-core must have at least 3 neighbors, a 3-core must contain at least 4 citizens. So adding a 14^{th} link creates a large jump in the fraction of citizens protesting—from zero, up to one half.

Adding further links has a 'concave' effect on the number of protesters. It is possible to add 2 more links while keeping the number of protesters to 5, to add 3 more links on top of that while keeping the number of protesters to 6, and then 4 more links on top of that while keeping the number of protesters to 7. Figure 2A shows this graphically.

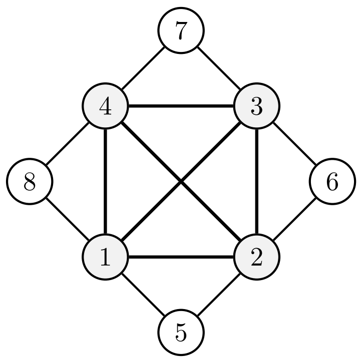
What does the government do? First note that the government would never want have more than 18 links here because the 'direct' productivity benefits $(36L - L^2)$ are maximized with 18 links. With 18 links, at least 6 citizens protest (see Figure 1D and Figure 2A).

We can easily calculate the government's payoff for different numbers of links. We show this in Figure 2B. The key insight is that it is never optimal to choose something just above the threshold (e.g., $L = 14, 15$). If protest is costly to the government ($\beta > \frac{100}{3}$) then it is best to stop *exactly* at 13 links. And if protest is not too costly to the government ($\beta < \frac{100}{3}$), it is best to go *far* past the threshold to the unconstrained optimum of 18 links. In this sense, there is a *missing middle* in the government's decision-making. Either it stops creating links before there is any protest, or it accepts significant protest and creates links far beyond the threshold. Intermediate numbers of links are never optimal; the government is "in for a penny, in for a pound."

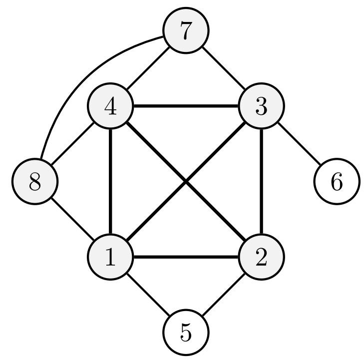
[7] This follows from Lick and White (Corollary 1, 1970).



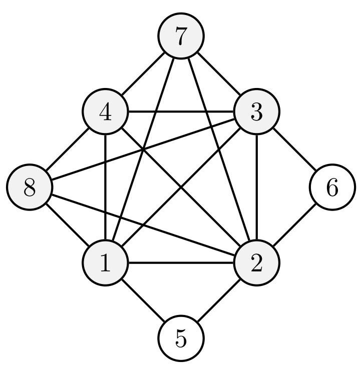
A. 13 links. No 3-core.



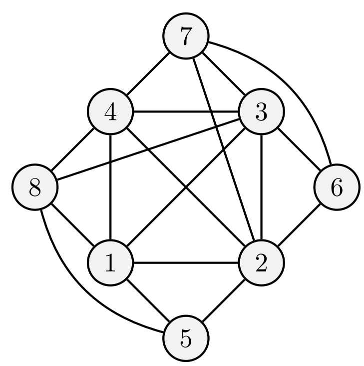
B. Adding link $3-4$ (14^{th} link) creates a 3-core: with citizens $\{1, 2, 3, 4\}$. Adding one further link (a 15^{th} link) must add a fifth citizen to the 3-core.



C. 14-link network with 6 protesters. Adding one further link (a 15^{th} link) can *remove* a citizen from the 3-core *if* rewiring is allowed.



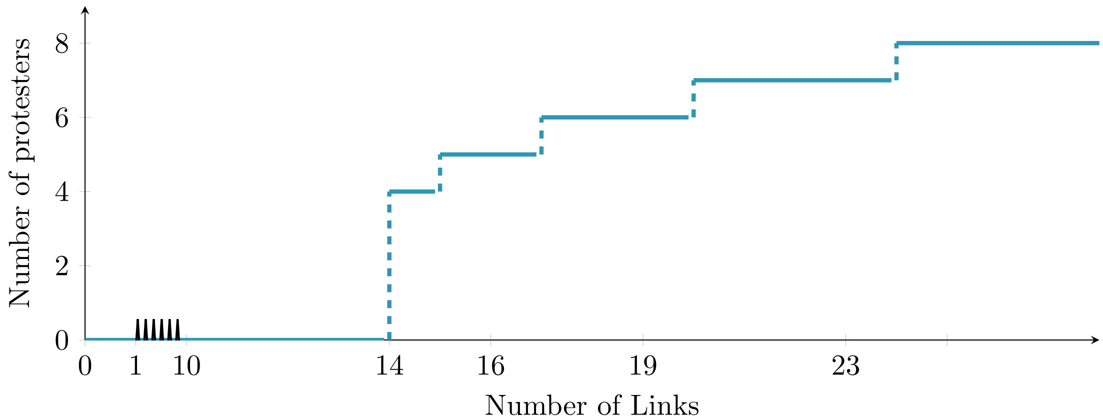
D. There is some protest, it is optimal to go well past the threshold; here there are 18 links, and 6 citizens are in the 3-core.



E. Optimal 19-link network if $k = 4$.

Figure 1

Limitation of fixed networks. On the face of it, equilibrium behavior appears to exhibit some regularities that depend on the underlying fixed network. For example, under the government's



A. Number of protesters is concave in number of links.

Links	≤ 13	14	15	16	17	18	19	20	21	22	23	≥ 24
Benefits	≤ 299	308	315	320	323	324	323	320	318	308	299	≤ 275
Costs	0	$\frac{4}{8}\beta$	$\frac{5}{8}\beta$	$\frac{5}{8}\beta$	$\frac{6}{8}\beta$	$\frac{6}{8}\beta$	$\frac{6}{8}\beta$	$\frac{7}{8}\beta$	$\frac{7}{8}\beta$	$\frac{7}{8}\beta$	$\frac{7}{8}\beta$	β

B. Benefits and costs by number of links.

Figure 2

optimal choice of network structure we observe concavity in the number of protesters as a function of the number of links (see Figure 2A). But making predictions about how protesting *changes* depends intimately on (1) the exactly structure of the network, and on (2) how much control we give the government.

On (1), we can see in Figure 1B that adding a link can either add one protester (e.g. link $8-3$) or two protesters (e.g. link $8-5$). But starting from the network in Figure 1C, adding a link can either add no protesters (e.g. link $8-3$) or one protester (e.g. link $8-5$).

On (2), whether the government can rewire completely the network, or only add links—which we contend is the conceptually preferable approach—matters a lot. In Figure 1C, if the government could rewire the network, then she could add one more link while reducing the number of protesters by one. But if she cannot, then adding one link to the existing network at best keeps the number of protesters constant.

A separate but related difficulty with analyzing fixed networks is that the 'best' network with a given number of links changes as the number of neighbors each citizen needs to protest for them to want to protest too (which we call the participation threshold, k) changes. But moving between optimal networks would require the principal to rewire links. The network in Figure 1D is optimal for the principal when $k = 3$—it has 6 protesters. But when k increases to 4, the number of protesters does not change. In stark contrast, the network in Figure 1E performs very poorly for the government when $k = 3$—all 8 citizens protest. But when k increases to 4, the number of protesters falls all the way to zero. So the microscopic details of the network structure preferred by the government depends critically on the participation threshold, k.

Taken together, these issues prevent clear comparative statics: we cannot say with any reasonable degree of precision what happens to the number of protesters as the connectivity or participation threshold change. This fundamental limitation is overcome by modeling the interaction structure as a large random network. Moreover, the random network setting lends itself more naturally to the constraints a government faces when designing a network in reality—that they cannot control its microstructure. We now present our formal model.

4 Model

We consider a sequence of games $\{\Gamma^{(n)}\}_{n\in\mathbb{N}}$ indexed by the number of agents n. We now describe the structure of a specific game $\Gamma^{(n)}$.

Agents, Actions & Timing. There is a single principal ('she'), P, and n agents ('he'), indexed $i \in N \equiv \{1, ..., n\}$. In the first period ($t = 1$), the principal chooses an *interaction rate* $r \geq 0$. Then Nature forms an Erdös-Rényi random graph $G(n, \frac{r}{n})$. That is, for any pair of agents, a link is formed independently with probability r/n. In a slight abuse of notation, we will let G denote the adjacency matrix of a realization of the random graph. So $G_{ij} = 1$ if i, j are linked (i.e. are neighbors) and $G_{ij} = 0$ otherwise.

In the second period ($t = 2$), each agent simultaneously chooses whether or not to take a binary action, $a_i \in \{0, 1\}$. For clarity, we will say that an agent who takes action 1 *participates*, and an agent who takes action 0 *abstains*.

Preferences: principal. We assume that the principal has a linear direct benefit from interactions among the agents, and faces a convex (for simplicity, quadratic) cost of increasing the interaction rate. Finally, participation by agents (denoted by $\bar{a}$) is costly to the principal. The principal's payoff function is:

$$\pi(r) = \alpha r - \tfrac{1}{2}r^2 - \beta\,\bar{a}, \tag{2}$$

where $\alpha, \beta > 0$, and $\bar{a} = \frac{1}{n}\sum_i a_i$ is the fraction of agents participating in the action.[8]

Preferences: agents. Agents' actions are *strategic complements*: participating is more attractive to agent i when more of her neighbors in the network participate. Specifically, an agent i prefers to participate (i.e., choose $a_i = 1$) if and only if at least $k \geq 3$ of her neighbors also participate. We call k the *participation threshold.* Agents' preferences can be represented by a utility function $u_i(a_i, M_i)$ such that:[9]

$$u_i\,(1, M_i) \geq u_i\,(0, M_i) \iff M_i \geq k, \qquad \text{where} \qquad M_i = \sum_{j \neq i} G_{ij} a_j. \tag{3}$$

Notice that because the network is unweighted, it is without loss to assume that k is an integer.

Information. For expositional simplicity, we assume that the realization of the network is common knowledge to agents. This means that actions a_i are functions $a_i = a_i(G)$ of the realized network $G = G(n, \frac{r}{n})$ observed by agents. In Online Appendix B we provide a microfoundation which demonstrates that our reduced-form model can be viewed as the outcome of a model in which agents participate in an explicit diffusion process and only observe the actions of their neighbors.

Equilibrium. A strategy profile $\left(r^*, a_1^*, a_2^*, \ldots, a_n^*\right)$ is a *subgame perfect Nash equilibrium* of the game $\Gamma^{(n)}$ if, for any realization of the graph $G(n, \frac{r^*}{n})$ the actions a_i^* constitute a Nash equilibrium

[8] We think of $\beta\,\bar{a}$ as capturing the overall harm to the principal from agents' participation. We could equally as well have assumed that the principal's payoff depends on some concave function of $\bar{a}$, and this would not qualitatively change our main results.

[9] An alternative payoff specification one might consider is one for which agents' thresholds depend on the *fraction* of their neighbors taking the action rather than the absolute number. We refer the reader to Watts (2002) for a canonical fractional-threshold model of cascades on random networks.

of the adoption phase:

$$\text{for all } i \in N, \quad a_i^* = 1 \iff u_i(1, M_i^*) \geq u_i(0, M_i^*),$$

and the firm chooses r^* such that

$$r^* \in \arg\max_{r \geq 0} \mathbb{E}_{\mathbb{P}_r}[\pi(r, \bar{a}^*)],$$

where $\bar{a}^* = \frac{1}{n}\sum_{i=1}^{n} a_i^*$ is the equilibrium level of participation in the second stage. The expectation $\mathbb{E}_{\mathbb{P}_r}$ is taken with respect to the distribution $\mathbb{P}_r$ over networks generated by the principal's choice of r. This definition of equilibrium thus requires that (i) agents are best-responding to other agents given the realized network, and (ii) the principal maximizes her expected payoff, anticipating both the network structure that arises from r and the resulting actions that agents choose.

4.1 Discussion

Having set out the formal model, it is helpful to briefly discuss some key assumptions: (i) actions by agents are strategic complements, (ii) the principal controls only the interaction rate, and (iii) the principal anticipates the "worst" equilibrium (the one with the highest level of participation by agents).

Strategic complementarities. The model's core mechanism is a participation threshold driven by strategic complementarities. We interpret this broadly as a process of social reinforcement, where an agent's incentive to act is fundamentally linked to the actions of their direct network connections. The parameter k represents the critical mass of local support required to make an action attractive.

Controlling the interaction rate. The assumption that the principal controls only the interaction rate captures a natural constraint: she can encourage (or discourage) agents from interacting, but cannot control exactly who they interact with. In our model, the principal lacks the fine-grained tools required to engineer the network exactly as she would like it. For example, if the principal were a firm and agents were workers, then our model takes the interactions among workers to be serendipitous. Interactions might provide a worker with the right piece of information at the right time, allowing them to solve a problem at hand or introduce a process innovation in their job. These types of interactions cannot easily be planned in advance.

Equilibrium Selection. Because the agents' actions exhibit strategic complementarities, the second stage of our game may admit multiple Nash equilibria (notably, $a_i = 0$ for all i is always a Nash equilibrium of the second stage). Given this multiplicity, our analysis focuses on the equilibrium with the *largest* number of participating agents.[10] This equilibrium represents a worst-case scenario for the principal. One natural reason for the principal to anticipate the largest equilibrium is if $a_i = 1$ represents the "default behavior", and $a_i = 0$ a "new" behavior. In this case, a standard adaptive best-response dynamic selects the largest equilibrium. We discuss this further in Online Appendix B.

4.2 Some Applications

To motivate the analysis, we briefly outline several settings that share the key features of our model. These examples are illustrative, not exhaustive; our formal results concern the general mechanism of complex contagion, which applies independently of any single empirical domain.

1. **Resisting Authoritarians.** A citizen's decision to oppose an authoritarian leader or political party often depends on what their friends do for several reasons: providing "safety in numbers", creating peer pressure to "do the right thing", and increasing the perceived impact of the action. A would-be authoritarian (the principal) can influence the interaction rate by limiting citizens' ability to gather, either in person or online. The empirical evidence is consistent with the view that regimes act on this lever, for example by targeting online connections along which behavior can spread (King et al., 2013) (and being much more permissive about connections that do not spread behavior),[11] or deploying all-or-nothing internet shutdowns

[10]Since, by definition, small equilibria can never be "viral" (in the sense of Sadler, 2020), several papers using games on random graphs have focused on the largest equilibrium (see, e.g. Campbell et al., 2024a).

[11]Note that our model could easily be extended to account for a multiplexed network in which other layers may

during protests and elections (Access Now, 2022; V-Dem Institute, 2024).

2. **Inclusive behavior at work.** Workers' decisions about whether to bully, harass, or shirk are often sustained by cumulative peer pressures: workers face pressures to do what their coworkers do. A firm (principal) that wants to change its workers' behavior from some existing default ($a_i = 1$) has significant—albeit indirect—influence over how much they interact: choosing where to put the water cooler, how to lay out the office, setting work-from-home policies, or implementing more direct measures like changing reporting structures or encouraging out-of-work socializing.

3. **New customs.** People's adoption of a new custom, fashion, or cultural norm often relies on peer pressure and "social proof". A new behavior feels more legitimate and less socially risky when one's friends have already adopted it. A firm or government that wants people to take up the new custom may be able to influence the social network—for example, by making the new custom more or less visible to a person's friends. In laboratory and field settings, adoption of social norms and conventions often displays tipping once a critical mass is reached (Centola, 2010; Centola et al., 2018), a qualitative pattern mirrored by the discontinuous onset of the giant k-core at c_k in our model.

Our model also describes many other settings, including the adoption of new technologies or the preservation of minority languages. With these applications in mind, we now characterize the behavior of the principal and the agents in equilibrium.

5 How Agents Behave

We begin with the second stage of the game, and characterize how agents' behavior depends on the interaction rate, r. The second stage is a threshold game played on the realized network, G. So before going further, it is important to discuss the role of the k-core in characterizing equilibrium behavior.

affect the payoffs of the principal (and/or of the agents) but do not exhibit strategic complementarities. Since these layers do not affect strategic incentives at the margin, they would have no impact on behavior.

5.1 Equilibrium and the k-core.

In threshold games on networks, the set of participating agents in the largest equilibrium is determined by the network's "core structure". The k-core of G is the largest subgraph H of G such that every agent in H has at least k neighbors in H (Seidman, 1983). In our model the set of agents who participate in the largest equilibrium is precisely the k-core of the network G. We formalize this in the following remark.

Remark 1. *In equilibrium, agent i participates ($a_i = 1$) if and only if they belong to the k-core of the network $G(n, \frac{r}{n})$.*

We can see why this is the case by considering how agents might reason about a stable outcome. Any agent with fewer than k neighbors in the entire network knows he can never meet the participation threshold, so he will choose to abstain. Knowing this, other agents can revise what they expect their neighbors to do. An agent who initially had k neighbors might now expect fewer than k to participate, causing him to also abstain. This iterated removal of agents who lack sufficient support continues until only a stable group remains. The agents left are exactly those in the k-core; each has at least k connections to others who are also participating.

The direct link between the largest equilibrium and the k-core provides us with a powerful analytical tool. The principal's problem can now be re-framed as choosing an interaction rate r to maximize her expected utility, knowing that the fraction of participants, $\bar{a}$, will be determined by the expected fraction of agents belonging to the k-core of the resulting random graph $G(n, r/n)$.

The relationship between equilibrium behavior and the k-core has been established in prior studies of threshold games on fixed (finite) networks (Gagnon and Goyal, 2017; Langtry et al., 2024). However, as illustrated in our introductory example (see Section 3), on fixed networks it is very difficult to make strong predictions about how equilibrium behavior changes as a function of the primitives of the models—comparative statics are elusive.For this reason, and in line with the literature on large-scale social and economic phenomena, the remainder of our analysis focuses on the asymptotic properties of the model as the number of agents $n \to \infty$. As such, the equilibria we characterize are limits of subgame perfect equilibria of the finite games $\Gamma^{(n)}$.[12] An important

[12] All references to "equilibrium" from hereon in should be taken to mean "limits of equilibria in the sequence of

feature of the limiting game will be the presence (or absence) of a *giant k-core:* a k-core containing a positive fraction of all agents (with high probability as $n \to \infty$). We will often omit the limit in our statement of results, but any such results should be understood as applying with high probability as $n \to \infty$. Our approach allows us to derive a sharp, analytical characterization of the equilibrium outcomes and to provide clear insights into the trade-offs the principal faces.

5.2 Equilibrium behavior

Our characterization of equilibrium behavior in the second stage relies on established results from random graph theory. The existence of a giant k-core is known to depend on a sharp cut-off condition (Pittel et al., 1996).[13] Above this cut-off, its size is strictly increasing in network connectivity—which, in our model, is the interaction rate r. This provides the crucial link between the principal's choice of r and the resulting equilibrium participation.

To fix ideas, we refer to the fraction of agents who choose action 1, $\overline{a}^* \equiv \lim_{n\to\infty} \frac{1}{n} \sum_i a_i^*$, as the (level of) *participation*. It is also convenient to define $\psi_k(r) = \mathbb{P}(\text{Poisson}(r) \geq k)$, and to let $\rho = \rho_k(r)$ be the largest solution in $[0, 1]$ to the equation

$$\rho = \psi_{k-1}(r\rho). \tag{4}$$

Equipped with these definitions, we can now characterize behavior in the second stage.

Theorem 1 (Participation)**.** *Fix $r \geq 0$ and define $c_k \equiv \min_{x \geq 0} \frac{x}{\psi_{k-1}(x)}$. Equilibrium participation:*

(i) is positive if and only if the interaction rate is sufficiently high ($r > c_k$).

(ii) is given by $\overline{a}_k^*(r) = \psi_k(r\rho)$, where ρ is as in Equation (4)*.*

Theorem 1 establishes that participation is a non-linear phenomenon. A crucial feature for $k \geq 3$ is that this emergence is *discontinuous*, in sharp contrast to the continuous emergence of the standard giant component (the $k = 2$ case).[14] Unless the interaction rate r exceeds the critical

finite games".

[13] The critical cut-off condition is usually referred to as the critical *threshold*, however to avoid confusion we reserve the word "threshold" for the participation threshold, k.

[14] Taking the giant component and removing all agents with degree 1 yields the 2-core. The critical cut-off for the emergence of the giant 2-core is identical to the critical cut-off for the emergence of the giant component ($r = 1$).

cut-off c_k, the network is too sparse to support a giant k-core, and equilibrium participation is zero. Above this cut-off, the fraction of participating agents $\overline{a}^*$ jumps from zero to a strictly positive value, characterized by part (ii).

The intuition for this discontinuity lies in the nature of social reinforcement required for participation. For any group of agents to form a stable, self-sustaining equilibrium (i.e., to be a k-core), each member must have their participation validated by at least k neighbors who are also part of that same group. When $k = 2$, this is a fairly weak requirement. When $k \geq 3$ it is far stronger. We provide a more detailed intuition for this in Online Appendix B. While the *existence* of this discontinuity is an established result in random graph theory, we provide a novel result on its *magnitude*. Importantly, we show that the size of the jump—which determines the minimum non-zero participation level the principal can induce—is economically important.[15]

Proposition 1 (Discontinuity)**.** *For all $k \geq 3$, if participation is positive then it is at least* 0.27*.*

Recent empirical work has identified a critical mass of approximately 25% as the "tipping point" required for a committed minority to overturn an established social norm (Centola et al., 2018). This finding highlights the necessary conditions for *initiating* a large-scale behavioral cascade. Proposition 1 provides a complementary theoretical perspective by characterizing the conditions for the resulting behavior to be self-sustaining: under a moderate participation threshold ($k = 3$), at least 27% of the population must participate.[16] Our result provides a possible explanation for this empirical tipping point: individuals are willing to adopt a new norm only when they perceive the movement has enough momentum to reach a self-sustaining state.

5.3 Comparative Statics: how agents respond to the environment.

We now turn to analyzing the properties of equilibrium participation as a function of the primitives: the interaction rate r and the participation threshold k. Part (ii) of Theorem 1 allows us to derive

[15] Since $\psi_k(r\rho)$ is an analytic function (see, e.g. Remark 4.7 Janson, 2009), the derivative ψ'_{k-1} exists and is well defined.

[16] When $k = 3$ the minimum nonzero participation is approximately 0.27, but as k increases, so does the minimum participation. For example, when $k = 10$, minimum participation is 0.74, an when $k = 100$ it is around 0.95. In the proof of Proposition 1 we provide a general expression for the minimum participation at any given k.

clear and intuitive comparative statics: the level of participation is increasing in the interaction rate and decreasing in the participation threshold.

The intuition here is straightforward. As the interaction rate increases, agents have more neighbors on average. This increases the likelihood that the number of neighbors of an agent who participates will exceed k, reinforcing participation. This is why the network can be a double-edged sword for the principal: higher interaction rates lead to more participation. Conversely, a higher threshold k means fewer agents will have sufficient participating neighbors, causing them to abstain. This initial abstention can then induce other agents to do the same, ultimately reducing participation.[17]

Beyond monotonicity, we show that equilibrium participation also exhibits diminishing returns. When participation positive, it is concave in both the interaction rate r and the threshold k. This creates interesting trade-offs for the principal which we explore in Section 7.

Theorem 2 (Comparative statics: monotonicity and concavity)**.**
Equilibrium participation, $\overline{a}_k^*(r)$, is:

(i) strictly increasing and strictly concave on (c_k, ∞) for every fixed k.

(ii) strictly decreasing and strictly concave in k for every fixed $r > c_{k+1}$. That is,

$$\overline{a}_{k+1}^*(r) - \overline{a}_k^*(r) < \overline{a}_k^*(r) - \overline{a}_{k-1}^*(r) < 0.$$

Immediately above the threshold a large measure of agents are "marginal"—each already has $k-1$ neighbors who would remain in the k-core once it forms, so a small increase in the interaction rate is very likely to give many of them the single extra link they need to qualify. Hence the derivative $\partial\bar{a}^*/\partial r$ is large just beyond c_k. As connectivity keeps rising, agents' degrees bifurcate: most agents either (i) possess far more than k core neighbors, making additional links redundant, or (ii) have very low total degree, so even an extra link or two still leaves them below the threshold. The mass of truly pivotal agents—those with exactly $k-1$ core neighbors—therefore shrinks monotonically

[17] In contrast, when there is zero participation in equilibrium, changes in the network connectivity and/or the participation threshold have no impact. This follows immediately from Theorem 1(i).

with r, and each subsequent increase in connectivity converts ever fewer new participants. The marginal return to r thus falls continuously, giving a strictly decreasing slope and hence a strictly concave participation curve.

This result shows that diminishing returns are not an ex-ante assumption, but rather an emergent property of the random network structure and the complex contagion process. While this concavity does not play a role in our characterization of the principal's behavior in Section 6,[18] it is useful for examining the effects of seeding (see 7). We also expect it may be independently useful in other models that consider complex contagion in a random network setting.

For example, consider an alternative version of our model where the principal chooses the participation threshold $k \geq 3$, rather than the interaction rate, r. If the costs (of setting k) were linear in k, there there would be extreme "bang-bang" solutions. Either the principal would do nothing, and leave k at 3. Or she would increase k to the point where there is zero participation.[19] This is a direct consequence of concavity. Since equilibrium participation $\overline{a}_k(r)$ is concave in k, its negative is convex. Hence a principal who *dislikes* participation and faces linear costs to increasing k maximizes a convex function against a linear cost, which naturally yields extreme solutions.

Additionally, both parts of Theorem 2 are new *mathematical* results in their own right. The concave relationship between the size of the giant k-core and each of network connectivity and the threshold, k—which is the technical step that underpins Theorem 2—was, to our knowledge, not previously known.

5.4 A summary in a picture

All of our results on the second-stage behavior can be seen graphically, by plotting network connectivity, r, against the size of the giant k-core for various values of k. Figure 3 does exactly this. We now turn to the first stage—the principal's equilibrium choice of r—taking as given how agents will behave in the second stage.

[18] This is because the principal dislikes participation in in our model, so subtracting a concave function (the participation) from another concave function (the direct benefits/costs) need not be either everywhere concave or everywhere convex.

[19] For any fixed $r > c_3$, it is clear that there exists a participation cutoff $\overline{k}(r)$ such that participation is positive if and only if $k \leq \overline{k}(r)$. The principal could reduce participation to 0 by increasing k above $\overline{k}(r)$.

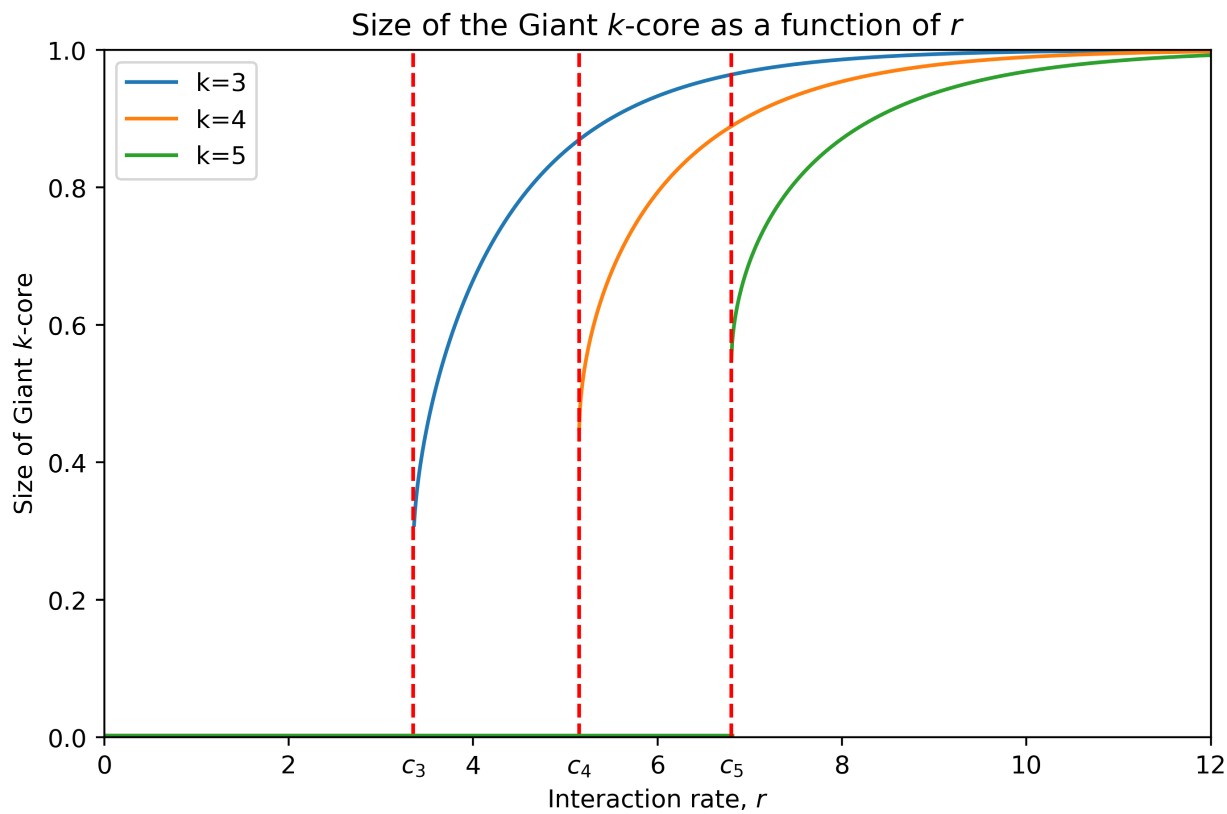


Figure 3: Size of the giant k-core as a function of network density, for $k = 3$ (blue) $k = 4$ (orange) and $k = 5$ (green). The red dashed lines show the critical cut-off for each value of k (Theorem 1). Each curve is strictly concave and increasing in the region where it is nonzero (Theorem 2(i)). Increasing k shifts the curve down, and where the giant k-core exists this has a larger impact at lower r (Theorem 2(ii)).

6 How the Principal Designs the Network

6.1 Choosing the interaction rate

In the first stage the principal chooses the interaction rate, r. She does so taking into account how agents will behave in the second stage. Using our characterization of second stage behavior in Theorem 1, the principal's payoff from choosing an interaction rate r is

$$\pi(r) = \alpha r - \frac{1}{2}r^2 - \beta\psi_k(r\rho(r)). \tag{5}$$

So the principal's problem is simple: choose the value of r that maximizes Equation (5). Since $\pi(0) = 0$, $\pi(r) < 0$ for sufficiently large r, and the participation function is continuous, an equilibrium must exist.[20] Moreover, when an equilibrium exists it is generically unique—if there are two or more equilibria then a small perturbation of β restores uniqueness (we discuss this further in Online Appendix B).

With existence and (generic) uniqueness in place, we now turn to a precise characterization of the equilibrium interaction rate r^*. We then show how r^* changes with primitives of the model. To do this, we leverage what we have proved about equilibrium participation in Section 5. Let $r^{\text{naive}} = \alpha$. This is the principal's optimal choice of r in the absence of any preference over agents' behavior (i.e., if $\beta = 0$). First, it is clear that the principal will never choose an interaction rate greater than r^{naive}. This is because any $r > r^{\text{naive}}$ is unambiguously worse: the benefit from interaction ($\alpha r - \frac{1}{2}r^2$) is already past its maximum, and the higher interaction rate *also* weakly increases costly participation.[21]

Second, the principal will never choose an interaction rate "just above" the critical cut-off. This is because participation drops discontinuously when the principal brings down the interaction rate exactly to the threshold. In contrast, the cost of doing so becomes smaller the closer the interaction rate is to the threshold. The result of this is a "missing middle" in the levels of network connectivity the principal might choose. She either goes all the way down to the critical cut-off, or stays at a significantly higher level.

To formalize this idea, it is convenient to define $\overline{\beta} = \overline{\beta}(\alpha) = \sup\{\beta\colon \sup_r \pi(r) \geq \alpha c_k - \frac{1}{2}c_k^2\}$. This is the highest weight (β) the principal can put on agents' participation for which she prefers positive participation in equilibrium. The following result uses $\overline{\beta}$ to precisely characterize the "missing middle". Figure 4 illustrates the intuition for the result.

Proposition 2 (The missing middle)**.** *Suppose the principal faces a nontrivial tradeoff ($c_k < \alpha$)*

[20] To be precise, observe that $\psi_k(r\rho(r)) \in [0, 1]$ for all r, k. So for r large enough, the $-r^2$ term in Equation (5) must dominate and hence $\pi(r) < 0$. This implies that there is is a compact interval $[0, b]$ such that $\pi(r) < 0$ for all $r > b$. Finally, since $\pi(r)$ is continuous it must attain a global maximum on $[0, b]$. Our analysis here relies on the fact that $\psi_k(r\rho(r))$ is continuous.

[21] This point implies that if $r^{\text{naive}} < c_k$, then the principal will always choose r^{naive}. Since our interest is in studying the cases where it is possible that the principal faces a trade-off between interaction and participation, we restrict attention in Proposition 2 to the case where $r^{\text{naive}} > c_k$.

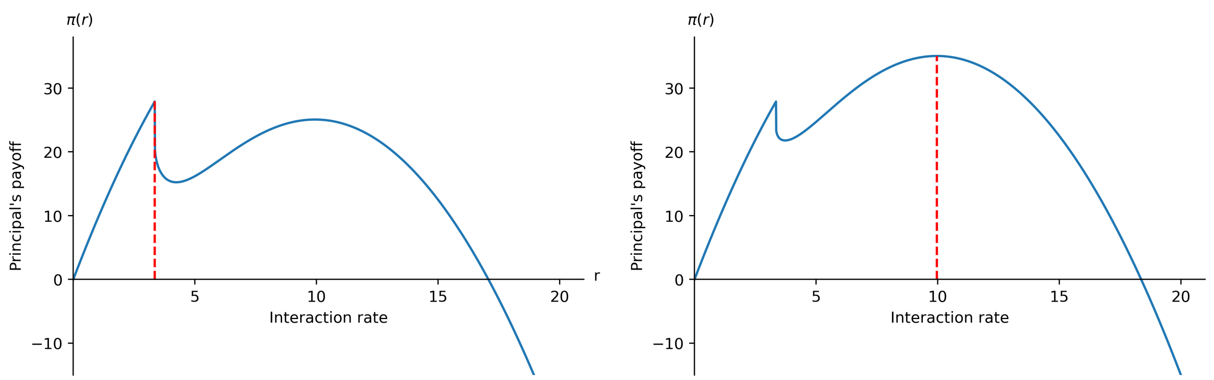
from participation.

(i) If participation is sufficiently costly ($\beta > \overline{\beta}$), then the principal chooses the highest interaction rate that does not induce participation ($r^* = c_k$).

(ii) If participation is sufficiently benign ($\beta < \overline{\beta}$), then the principal chooses an interaction rate at least $d_k > 0$ above the critical threshold, where $d_k = \frac{1}{2} \cdot \frac{\beta\, \psi_k(x_k^\star)}{\alpha - c_k} > 0$.

Intuitively, if the principal is going to allow nonzero participation in equilibrium, then she must allow enough to recuperate their losses from moving beyond the critical cut-off—that is, from allowing it at all. This is what leads to the missing middle. The term d_k is then a lower bound on how far the principal must move up from the critical cut-off to recover her losses.

The missing middle also creates a sensitivity in the principal's optimal choice of r. Although any equilibrium with positive participation has *a lot* of participation, small changes in the incentives (α or β) may induce the principal to "jump" from an equilibrium with substantial participation down to one with zero (or vice versa). To explore this sensitivity, we first describe how the principal's equilibrium choice of r depends on these parameters.



Case (i) $\beta > \overline{\beta}$ Case (ii) $\beta < \overline{\beta}$

Figure 4: The principal's equilibrium choice of interaction rate, r. With $k = 3$, $\alpha = 10$. $\beta = 25$ in left panel, and $\beta = 15$ in right panel.

Comparative Statics. Recall that the principal's payoff is governed by two parameters: α, her marginal benefit from agents' interactions; and β, her marginal cost from agents' participation.

A key implication of Proposition 2 is that the principal's responsiveness to changes in her objectives is not constant. The effect of a change in the cost of participation (β) depends crucially on whether her optimal strategy is to suppress participation entirely ($\beta > \overline{\beta}$), or to tolerate it ($\beta < \overline{\beta}$). The following proposition formalizes this sensitivity.[22]

Proposition 3 (Comparative Statics and Tipping Points)**.** *For any fixed $\alpha > 0$ such that $c_k < \alpha$,*

(i) If participation is sufficiently costly ($\beta > \overline{\beta}$), equilibrium participation is not responsive to changes in β ($\frac{d\overline{a}^*}{d\beta} = 0$).

(ii) If participation is sufficiently benign ($\beta < \overline{\beta}$), equilibrium participation is strictly decreasing in β ($\frac{d\overline{a}^*}{d\beta} < 0$).

An analogous threshold $\overline{\alpha}(\beta)$ exists.

Proposition 3 highlights two distinct regimes. When the principal tolerates the behavior in equilibrium ($\beta < \overline{\beta}$), their adjustments are smooth: as β increases, the principal reduces the interaction rate. This, in turn, leads to a reduction in participation.

In contrast, when the optimal strategy is to completely suppress participation ($\beta > \overline{\beta}$), the principal's choice of r^* is "sticky". Small changes in her incentives are insufficient to move the interaction rate away from the critical cut-off; she continues to choose the highest possible interaction rate that guarantees zero participation.

The key takeaway from Propositions 2 and 3 is that when β is close to $\overline{\beta}$, a small change in the cost of participation can move her from choosing an equilibrium with positive participation to one with zero participation (or the reverse). This highlights a tipping-point phenomenon in the principal's decision-making: her strategy, and more importantly the resulting participation, can shift dramatically in response to relatively mild changes in her incentives.

[22] We could equally have defined a critical threshold for α, and stated both Propositions 2 and 3 with respect to it. For simplicity and consistency we have stated these results with respect to β as it is the most "novel" parameter of the model.

7 Seeding

As a final exercise, we consider what happens when a third party intervenes by "seeding" the network. Here, we think about seeding "good apples"—that is converting agents into committed abstainers, who will not participate no matter what their friends do.[23] As a benchmark, we assume that seeding converts agents into good apples independently with probability $\sigma \in (0, 1)$. One natural interpretation here is that the third party *tries* to convert all agents into good apples, and succeeds independently with probability σ. Viewed this way, we interpret σ as the *effectiveness* of the intervention.

The impact that seeding has on the equilibrium level of participation depends critically on whether it is anticipated by the principal. If it not anticipated, then it will reduce participation significantly. If it *is* anticipated, then the principal will respond by increasing the interaction rate. We show that this still results in a decrease in the overall participation, but by much less than with unanticipated seeding. This distinction between anticipated and unanticipated seeding is important for a third party who engages in the seeding with the aim of reducing participation. We now cover each setting in turn.

Unanticipated seeding. This intervention has two distinct effects on equilibrium participation. First, there is a direct effect: converting a fraction σ of agents into good apples mechanically decreases participation by that fraction. Second, and more interestingly, there is an indirect network effect. Agents who are connected to good apples now have fewer participating neighbors, which may push them below the participation threshold. This can trigger cascading abstention: agents who stop participating due to their converted neighbors may in turn influence their own neighbors to stop. The interplay between these direct and indirect effects depends crucially on the network structure. In sparse networks (when r is small), converted agents may be isolated from one another, limiting the potential for cascades. In denser networks, the same fraction of converted agents can trigger widespread changes in behavior through reinforcing local spillovers. Whether seeding

[23] One way of thinking about this is as changing the participation threshold for these agents to $+\infty$. We use "seeding" in the same sense as, e.g.Akbarpour et al. (2020); Sadler (2020), but here the seeded action is $a = 0$ rather than $a = 1$.

eliminates participation depends entirely on its effectiveness.

Proposition 4. *Fix $r \geq 0$ and define $\sigma^{crit} \equiv 1 - \frac{c_k}{r}$. Then,*

(i) Participation is eliminated after the intervention if and only if $\sigma \geq \sigma^{crit}$.

(ii) The equilibrium participation after converting σ good apples is given by

$$\overline{a}^{\text{seed}}_{\sigma}(r) \equiv \underbrace{(1-\sigma)}_{\textit{direct effect}} \cdot \overline{a}^*(\underbrace{(1-\sigma)r}_{\textit{indirect effect}})$$

where $\overline{a}^*$ is the baseline equilibrium participation from Theorem 1.

Notably, there exists a critical threshold σ^{crit} above which the intervention eliminates participation—a discontinuous jump. Because r^* is large whenever it is above c_k (recall the "missing middle" of Proposition 2), a large number of good apples is needed to eliminate participation. However, since σ^{crit} is decreasing in k a higher participation cutoff implies that participation is easier to dismantle. We discuss the importance of this further in Section 7.1.

Anticipated seeding When the principal anticipates seeding, she increases the interaction rate, r, relative to what it would have been in if seeding were unanticipated.[24] Both the direct and indirect effects identified in Proposition 4 cause (a) a reduction in the level of participation, $\overline{a}^*$, for any given interaction rate, r, and (b) a "dampening" of the strength of the relationship between the two—a given increase in r results in a smaller increase in $\overline{a}^*$. As a result, more effective seeding (higher σ) decreases the marginal cost to the principal from increasing the interaction rate, since the resulting uptake in participation is not as large as with zero seeding. This is why the optimal interaction rate increases in the effectiveness of seeding.

However, it is not immediately clear whether this increase in the principal's choice of interaction rate overwhelms the direct and indirect effects which work to decrease participation. We show that the equilibrium participation is decreasing in the effectiveness of seeding, σ, even when the seeding

[24]From the principal's perspective, unanticipated seeding is identical to anticipated seeding with $\sigma = 0$. That is, the principal chooses r as though there is no seeding.

is anticipated. Write $r = r^*_\sigma$ for the principal's optimal choice of interaction rate given that she anticipates seeding σ.

Proposition 5. *If equilibrium participation is nonzero ($r^*_\sigma > c_k$), then it is strictly decreasing in σ.*

Although equilibrium participation is decreasing in σ, it is worth noting that this reduction is muted relative to unanticipated seeding. This is because although overall participation decreases, the principal still raises the interaction rate. One implication of this is that a third party who wants to reduce the level of participation would have an incentive to conceal their intention of seeding so that it is unanticipated by the principal.

7.1 The effectiveness of random seeding.

Complex Contagion. The analysis in this section focuses on seeding to *reduce* the level of participation, starting from the maximal equilibrium. A consequence of the analysis is that random seeding is not very effective for two reasons: (a) seeding a negligible fraction of all agents has a negligible impact, and (b) highly targeted seeding is much more effective than random seeding.[25]

In a similar complex contagion setting, Jackson and Storms (forthcoming) consider seeding to *increase* the level of participation, starting from the minimal equilibrium (where nobody takes the action $a_i = 1$). They also find that random seeding is not very effective, in both of the senses above. Further, Jackson and Storms (forthcoming, esp. S4) provide an algorithm that *is* effective.[26]

Simple Contagion. Akbarpour et al. (2020) also consider seeding to *increase* the level of participation, starting from the minimal equilibrium (where nobody takes the action $a_i = 1$), but do so in a *simple contagion* environment (in their setting, an agent will take an action if at least one of his neighbors does so). Importantly, they highlight that (a) randomly seeding even a finite number of agents can result in a high level of participation, and (b) random seeding with only a few extra seeds often outperforms highly targeted seeding strategies in this setting. Relatedly, Sadler (2025)

[25] It is clear from our model that the modest refinement of seeding only agents who are in the k-core (i.e. "targeted seeding") is significantly more effective than seeding at random.

[26] The question of optimal seeding in these environments has attracted significant attention, see for example Kempe et al. (2003, 2005); Mossel and Roch (2010), and closer to our setting, Schoenebeck et al. (2022).

shows that in simple contagion, targeted seeding delivers gains primarily in the subcritical (non-viral) regime; once the network is viral (i.e. density is past the critical cut-off), random seeding effectively reaches the giant component and targeting offers only modest improvements among peripheral, often lower-degree, nodes. Intuitively, this is because participation spread from each seed to all agents who are indirectly connected to the seed. So reaching even a single seed who belongs to a large component of the network will have a large impact on the overall level of participation.

In stark contrast, in a simple contagion setting random seeding will be ineffective when trying to *reduce* the level of participation (starting from the maximal equilibrium). This is because in the maximal equilibrium, all agents who have at least one friend will participate. So reducing participation from the largest equilibrium requires either seeding an agent directly, or completely isolating them by seeding all of their neighbors. This means that (a) a small number of random seeds will have a small impact, and (b) carefully targeted seeding can perform significantly better than random seeding (e.g. targeting "hubs" of local star-like sub-networks to isolate many individuals).

Comparing cases. This highlights an important asymmetry between simple and complex contagion environments when it comes to the efficacy of seeding. In a complex contagion environment random seeding is never very effective—regardless of whether the seeding is trying to increase or reduce participation. This is true both in the sense that (a) a few random seeds have little effect and (b) careful targeting does much better.

In contrast, the efficacy of random seeding in a *simple* contagion environment depends critically on whether the aim is to increase or reduce participation. When trying to increase participation, random seeding is highly effective. But when trying to reduce participation, it is *even less* effective than in complex contagions. Again, this is true both in the sense of (a) the impact that a few random seeds have and (b) the incremental benefits of careful targeting.

8 Conclusion

This paper provides a tractable framework for analyzing complex contagions by leveraging the theory of random graphs. This approach delivers a sharp, analytical characterization of equilibrium behavior and associated comparative statics—a task that has been intractable in fixed-network settings. We show that equilibrium participation is governed by the emergence of the giant k-core, an object well-studied in mathematics but whose economic implications have remained largely unexplored. Our primary technical contributions—in particular the concavity of the k-core (Theorem 2)—are, to our knowledge, new results that provide the foundation for our economic analysis.

Our model allows us to capture the subtle ways that agents' behavior responds to network connectivity. First, when connectivity is below a critical cut-off, there is no participation at all. Participation jumps discontinuously just above that cut-off, and then grows in a concave way as connectivity rises further. The discontinuity implies a minimum critical mass for any behavior to be self-sustaining, a finding we connect to recent empirical work on social norm "tipping points". The concavity implies diminishing effects of raising the interaction rate on participation—a key economic property that emerges as a direct consequence of the network structure.

Together, these properties of equilibrium participation generate a "missing middle" in the principal's optimal choice of network connectivity. We show she will either choose an interaction rate (equal to the critical cutoff) that guarantees zero participation or a high one that induces substantial participation, but never something in between. This, in turn, makes her optimal strategy fragile: a small change in her incentives can "tip" her from an equilibrium with zero participation to one with a large, positive fraction of participants.

We adopted a deliberately parsimonious framework to isolate these core mechanisms. As such, our model abstracts from many rich features of real-world networks. These features include homophily—the tendency for people to form links with those similar to themselves—and multiplexity, the idea that there can be different types of links that play different roles. Additionally, agents in our framework do not make endogenous decisions about forming or severing links; the network structure is determined solely by the principal's choice of interaction rate and the resulting realization of the random network. In reality, agents may have some control over both how many links to form, and who to form them with. Finally, we assume that agents are ex-ante homogeneous, though their positions (and degrees) within the network differ after its formation. This was a deliberate choice to isolate the role of the network rather than have our results be driven by heterogeneous preferences.

Recognizing these simplifications, we believe these avenues provide a clear agenda for future research to help better understand complex contagions. Some extensions to our framework are straightforward; for example, it is relatively simple to extend our model to allow for heterogeneous participation cutoffs. Others, however, represent a more significant challenge. Incorporating features like homophily or endogenous network formation into a tractable random graphs model is particularly difficult, but remains an important open question. Despite these abstractions, our framework's tractability offers a distinct advantage: its conclusions do not require detailed data on the entire network structure. Instead, the key trade-offs are driven by a single, aggregate parameter: the interaction rate. This should serve as a useful guiding principle for future empirical and theoretical work seeking to incorporate more sophisticated network features.

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A Proofs

A.1 Notation

It is convenient to begin by fixing notation and stating a number of useful equalities that are used repeatedly in the remainder of the appendix.

For integers $k \geq 3$ and $r > 0$, let

$$\psi_k(x) \equiv \mathbb{P}(\text{Poisson}(x) \geq k), \qquad p_k(x) \equiv \mathbb{P}(\text{Poisson}(x) = k) = e^{-x}\frac{x^k}{k!}.$$

Let $\rho_k(r) \in (0, 1]$ be the largest solution of $\rho = \psi_{k-1}(r\rho)$ and set

$$x_k(r) \equiv r\rho_k(r), \qquad S_k(r) \equiv \psi_k(x_k(r)).$$

The definition of $x_k(r)$ and $\rho_k(r)$ imply

$$\psi_{k-1}(x_k) = \frac{x_k(r)}{r}. \tag{A.1}$$

We will use the following standard identities (valid for $x > 0$ and all $k \geq 1$):

$$\psi_k'(x) = p_{k-1}(x), \tag{A.2}$$

$$\psi_{k-1}(x) = \psi_k(x) + p_{k-1}(x), \tag{A.3}$$

$$p_k(x) = \frac{x}{k}\, p_{k-1}(x), \tag{A.4}$$

$$p_k'(x) = p_{k-1}(x) - p_k(x). \tag{A.5}$$

Applying A.3 and A.1 to the definition of $S_k(r)$ gives

$$S_k(r) = \psi_k(x_k) = \psi_{k-1}(x_k) - p_{k-1}(x_k) = \frac{x_k}{r} - p_{k-1}(x_k). \tag{A.6}$$

A.2 Proof of Theorem 1

We first state the following theorem.

Theorem A.1 (Pittel et al. (1996))**.** *Let $\psi_k(r), \rho_k(r)$, and c_k be as in Appendix A.1.*

(i) The network $G(n, r/n)$ contains a giant k-core with high probability if and only if $r > c_k$,

(ii) The size of the giant k-core is given by $\psi_k(r\rho_k(r))$.

Since the largest equilibrium in the limit is one in which agents participate if and only if they are in a k-core (by Remark 1), it follows that a positive fraction of agents participate if and only if $G(n, r/n)$ has a giant k-core. Theorem 1 follows immediately. □

A.3 Proof of Proposition 1

Let $x_k^\star = \arg\min_{x \geq 0} \frac{x}{\psi_{k-1}(x)}$ so that $c_k = \frac{x_k^\star}{\psi_{k-1}(x_k^\star)}$. It is easily seen that $x_k^\star = x_k(c_k) = c_k \rho_k(c_k)$. Here we prove that at the critical cutoff c_k, the size of the giant k-core is given by

$$S_k(c_k) = \psi_k'(x_k^\star)(k-2).$$

The result then follows immediately from the fact that the k-core is strictly increasing in r for all $r \geq c_k$ (see Theorem 2).

First, observe that by the first-order condition for $x_k^\star$, we have

$$\psi_{k-1}(x_k^\star) - x_k^\star \psi_{k-1}'(x_k^\star) = 0 \implies \psi_{k-1}(x_k^\star) = x_k^\star \psi_{k-1}'(x_k). \tag{A.7}$$

Hence

$$\begin{aligned}
S_k(c_k) &= \psi_{k-1}(x_k^\star) - p_{k-1}(x_k^\star) && \text{(By A.6)}\\
&= x_k^\star \psi_{k-1}'(x_k^\star) - p_{k-1}(x_k^\star) && \text{(By A.7)}\\
&= x_k^\star \psi_{k-1}'(x_k^\star) - \psi_k'(x_k^\star) && \text{(By A.2)}\\
&= (k-1)\psi_k'(x_k^\star) - \psi_k'(x_k^\star) && \text{(By A.4)}\\
&= (k-2)\psi_k'(x_k^\star),
\end{aligned}$$

which completes the proof. It can be verified numerically that $\psi_3'(x_3^\star) > .27$ which gives the lower bound found in the proposition. □

A.4 Proof of Theorem 2

This is by far the most involved proof in the entire appendix. To make things easier for the reader, we divide the proof into two parts: first, the proof of concavity in r, second, the proof of (discrete) concavity in k.

A.4.1 Preliminary Lemmas

We now state and prove four useful lemmas.

Lemma A.1 (Downward propagation in k)**.** *$x_{k+1}(r) \leq x_k(r)$ for all k, with strict inequality if $x_k(r) > 0$.*

Proof. Fix $r > 0$ and $k \geq 3$. Define $g_k(x) \equiv r\,\psi_{k-1}(x) - x$ for $k \geq 3$. Observe that by definition, $x_k(r)$ is the largest zero of g_k. Hence $g_k(x) \leq 0$ for all $x \geq x_k(r)$. Moreover, for every $x > 0$,

$$g_{k+1}(x) = r\,\psi_k(x) - x \overset{A.3}{=} r\big(\psi_{k-1}(x) - p_{k-1}(x)\big) - x = g_k(x) - r\,p_{k-1}(x) \leq g_k(x),$$

with strict inequality for $x > 0$ since $p_{k-1}(x) > 0$. Since $g_{k+1} \le g_k$ and both functions ar continuous, it follows that $x_k(r) = 0$ implies $x_{k+1}(r) = 0$. Suppose now that $x_k(r) > 0$.

Evaluating g_{k+1} at $x_k(r)$ gives

$$g_{k+1}(x_k(r)) = g_k(x_k(r)) - rp_{k-1}(x_k(r)) = -rp_{k-1}(x_k(r)) < 0.$$

Hence g_{k+1} has no zero in $[x_k(r), \infty)$, so its largest zero $x_{k+1}(r)$ satisfies $x_{k+1}(r) < x_k(r)$. $\square$

Lemma A.2 (Stability at the largest root)**.** *At $x_k(r)$ we have $r\, p_{k-1}\big(x_k(r)\big) \le 1$, with strict inequality if $r > c_k$.*

Proof. Let $k \ge 3$ and define $h_k(x) \equiv r\,\psi_{k-1}(x) - x$. Then $x_k(r)$ is the largest zero of h_k and

$$h_k'(x) = r\,\psi_{k-1}'(x) - 1 \overset{A.2}{=} r\,p_{k-2}(x) - 1.$$

Because $h_k(x) \le 0$ for all $x \ge x_k(r)$ (largest zero), we must have $h_k'(x_k(r)) \le 0$, i.e.

$$r\,p_{k-2}\big(x_k(r)\big) \le 1.$$

Moreover, equality holds if and only if $x_k(r)$ is a tangential (double) root, which corresponds precisely to $r = c_k$ (the minimizer level of $x/\psi_{k-1}(x)$). Thus for $r > c_k$, the inequality is strict.

Remark (index shift). The analogous statement at level $k+1$ is $r\,p_{k-1}\big(x_{k+1}(r)\big) \le 1$, with strict inequality for $r > c_{k+1}$. This version is obtained by applying the same argument to $h_{k+1}(x) = r\psi_k(x) - x$. $\square$

Lemma A.3 (Unimodality of p_k)**.** *p_k increases on $(0, k]$ and decreases on $[k, \infty)$.*

Proof. From (A.5) we have

$$p_k'(x) = p_{k-1}(x) - p_k(x) \overset{A.4}{=} p_k(x)\Big(\frac{k}{x} - 1\Big).$$

Thus $p_k'(x) > 0$ for $x \in (0, k)$, $p_k'(k) = 0$, and $p_k'(x) < 0$ for $x > k$. Hence p_k increases on $(0, k]$ and decreases on $[k, \infty)$. $\square$

Lemma A.4 (Lower bound on x_k)**.** *For all $k \ge 4$, if $r \ge c_k$, then $x_k(r) > k - \frac{3}{4}$.*

Proof. Let $k \ge 4$. Recall the notation $x_k^\star = \arg\min_{x \ge 0} \frac{x}{\psi_{k-1}(x)}$ (as used in Proposition 1. Since $x_k(r) = r\psi_k(x_k(r))$ is the largest solution to $r = \frac{x}{\psi_{k-1}(x)}$ it must be greater than the minimizer $x_k^\star$. Hence it suffices to prove that $x_k^\star > k - \frac{3}{4}$ for all $k \ge 4$.

Now, $x_k^\star$ must satisfy the first order condition

$$\frac{d}{dx}\left(\frac{x}{\psi_{k-1}(x)}\right)\Bigg|_{x = x_k^\star} = 0.$$

The derivative on the left hand side is given by

$$\frac{\psi_{k-1}(x) - x\psi'_{k-1}(x)}{\psi_{k-1}(x)^2},$$

and this derivative is negative to the left of $x_k^\star$. Hence it suffices to prove that

$$\psi_{k-1}(\lambda) < \lambda\psi'_{k-1}(\lambda),$$

where $\lambda = k - \frac{3}{4}$. To this end, define

$$r_m = \frac{\mathbb{P}(\mathrm{Po}(\lambda) = k+m-2)}{\mathbb{P}(\mathrm{Po}(\lambda) = k-2)} = \frac{\lambda^m}{(k+m-2)(k+m-3)\ldots k(k-1)},$$

so that it suffices to prove

$$\sum_{m=1}^{\infty} r_m < \lambda.$$

For any $m \geq 3$, we have

$$\frac{r_{m+1}}{r_m} = \frac{\lambda}{k+m-1} \leq \frac{\lambda}{k+2},$$

and since $\lambda = k - \frac{3}{4}$, we have $\frac{\lambda}{k+2} < 1$. Moreover,

$$\frac{1}{1 - \frac{\lambda}{k+2}} = \frac{k+2}{k+2-(k-\frac{3}{4})} = \frac{4(k+2)}{11}.$$

Hence

$$\begin{aligned}
\sum_{m=1}^{\infty} r_m &\leq r_1 + r_2 + r_3\left(1 + \frac{\lambda}{k+1} + \left(\frac{\lambda}{k+1}\right)^2 + \ldots\right) \\
&= \frac{\lambda}{k-1} + \frac{\lambda^2}{k(k-1)} + \frac{\lambda^3}{k(k-1)(k+1)} \cdot \frac{1}{1 - \frac{\lambda}{k+2}} \\
&= \frac{\lambda}{k-1} + \frac{\lambda^2}{k(k-1)} + \frac{4\lambda^3(k+2)}{11k(k-1)(k+1)}.
\end{aligned}$$

So $\sum_{m=1}^{\infty} r_m \leq \lambda$ is satisfied if

$$\frac{\lambda}{k-1} + \frac{\lambda^2}{k(k-1)} + \frac{4\lambda^3(k+2)}{11k(k-1)(k+1)} < \lambda \iff \frac{4(k+2)\lambda^2}{11(k+1)} + \lambda - k(k-2) < 0.$$

Substituting $\lambda = k - \frac{3}{4}$, we find that the resulting cubic inequality is satisfied for all $k \geq 3.938$, and in particular this implies the lemma holds for all $k \geq 4$, completing the proof. $\square$

Lemma A.5 (Integral bound for decreasing functions)**.** *Let $f\colon [a,c] \to [0,\infty)$ be decreasing and* *let $b \in (a,c)$. Then*

$$\int_b^c f(x)\,dx \le \frac{c-b}{b-a}\int_a^b f(x)\,dx.$$

Proof. Since f is decreasing,

$$\int_b^c f(x)\,dx \le (c-b)f(b), \quad \text{and} \quad \int_a^b f(x)\,dx \ge (b-a)f(b).$$

Putting these inequalities together gives

$$\int_b^c f(x)\,dx \le (c-b)f(b) \le \frac{c-b}{b-a}\int_a^b f(x)\,dx,$$

as claimed. □

A.4.2 Proof: Participation is increasing and concave in r (part (i))

Suppose $r > c_k$ is the interaction rate so that a non-empty k-core exists with high probability as $n \to \infty$. Recall that we write $S_k(r) = \psi_k(x_k(r))$ for the asymptotic fraction of vertices in the k-core. We will show that $S_k''(r) < 0$ for all $k \ge 3$. First, we derive an expression for $S_k''(r)$. During this step we also show $S_k'(r) > 0$, which proves that S_k is strictly increasing in r. Second, we use this expression to simplify the conditions required for $S_k''(r) < 0$. Finally, we establish the result.[27]

Step 1: An expression for $S_k''(r)$. For notational convenience write $p_j = p_j(x_j(r))$ as defined in Appendix A.1. Implicitly differentiating $x_k(r) = \psi_{k-1}(x_k(r))r$ and using A.2, we find that

$$x_k'(r) = \psi_{k-1}(x_k(r)) + rp_{k-2}x_k'(r) \tag{A.8}$$

and therefore

$$x_k'(r) = \frac{\psi_{k-1}(x_k(r))}{1 - rp_{k-2}}. \tag{A.9}$$

It follows from Lemma A.2 that the denominator—and hence x_k'—is positive. Next, substituting A.1 into Equation (A.9) gives

$$x_k(r) = x_k'(r)r(1 - rp_{k-2}). \tag{A.10}$$

This equation will be useful later on in the proof. Another useful equation which follows from A.5 is

$$\frac{dp_k}{dr} = x_k'(r)(p_{k-1} - p_k). \tag{A.11}$$

[27]We are grateful to Krishna Dasaratha for sharing the idea for this proof with us. Krishna proved the concavity of the k-core for $k \ge 12$, and by adapting his argument we were able to prove the case for $k \ge 3$.

Next, we compute the first derivative $S_k'(r)$. By the chain rule, $S_k'(r) = x_k'(r)\psi_k'(x_k(r))$. Substituting $\rho = x_k(r)/r$, into eq. (A.9) and using the fact that $\psi_k'(x_k(r)) = p_{k-1}$, we have

$$\begin{aligned} S_k'(r) &= \frac{x_k(r)p_{k-1}}{r - r^2 p_{k-2}} \\ &= \frac{kp_k}{r - r^2 p_{k-2}} && \text{(by A.4)} \end{aligned}$$

Again by Lemma A.2, the denominator—and hence S_k'—is strictly positive. This proves that equilibrium participation is strictly increasing in the interaction rate r.

Differentiating again,

$$S_k''(r) = k \cdot \frac{x_k'(r)(p_{k-1} - p_k)(r - r^2 p_{k-2}) - p_k + 2rp_k p_{k-2} + x_k'(r) r^2 (p_k p_{k-3} - p_k p_{k-2})}{(r - r^2 p_{k-2})^2}$$

where we have used A.11.

Step 2: Simplifying conditions for $S_k''(r) < 0$. We now simplify the conditions required to prove that $S_k''(r) < 0$. The expression for $S_k''(r)$ has the same sign as

$$x_k'(r)(p_{k-1} - p_k)(r - r^2 p_{k-2}) - p_k + 2rp_k p_{k-2} + x_k'(r) r^2 (p_k p_{k-3} - p_k p_{k-2}).$$

This simplifies to

$$x_k'(r)(r(p_{k-1} - p_k) + r^2(p_k p_{k-3} - p_{k-1}p_{k-2})) - p_k + 2rp_k p_{k-2}. \tag{A.12}$$

We claim first that

$$x_k'(r) r^2 (p_k p_{k-3} - p_{k-1} p_{k-2}) + 2rp_k p_{k-2} = \frac{(rp_{k-2})^2 p_{k-1} r x_k'(r)}{k}.$$

The left-hand side is

$$x_k'(r) r^2 e^{-2x_k(r)} \left[\frac{x_k(r)^k}{k!} \cdot \frac{x_k(r)^{k-3}}{(k-3)!} - \frac{x_k(r)^{k-2}}{(k-2)!} \cdot \frac{x_k(r)^{k-1}}{(k-1)!} \right] + 2\lambda e^{-2x_k(r)} \left[\frac{x_k(r)^k}{k!} \cdot \frac{x_k(r)^{k-2}}{(k-2)!} \right]$$

$$= \frac{e^{-2x_k(r)} x_k(r)^{2k-3}}{(k-1)!(k-3)!} \left[\frac{r^2 x_k'(r)}{k} - \frac{r^2 x_k'(r)}{k-2} + \frac{2rx_k(r)}{k(k-2)} \right].$$

Substituting eq. (A.10) for $x_k(r)$, we can write this as

$$\begin{aligned}
&\frac{e^{-2x_k(r)}x_k(r)^{2k-3}r^2x_k'(r)}{(k-1)!(k-3)!}\left[\frac{1}{k}-\frac{1}{k-2}+\frac{2(1-rp_{k-2})}{k(k-2)}\right] \\
&=\frac{e^{-2x_k(r)}x_k(r)^{2k-3}r^2x_k'(r)}{(k-1)!(k-3)!}\left[-\frac{2}{k(k-2)}+\frac{2(1-rp_{k-2})}{k(k-2)}\right] \\
&=-\frac{e^{-2x_k(r)}x_k(r)^{2k-3}r^3x_k'(r)p_{k-2}}{k!(k-2)!}.
\end{aligned}$$

Expanding p_{k-2} and rearranging terms, we can further simplify this as

$$\begin{aligned}
-\frac{e^{-2x_k(r)}x_k(r)^{2k-4}r^2}{k!(k-2)!}\cdot\frac{e^{-x_k(r)}x_k(r)^{k-2}x_k(r)rx_k'(r)}{(k-2)!} &= -\left(\frac{e^{-x_k(r)}x_k(r)^{k-2}r}{(k-2)!}\right)^2\frac{e^{-x_k(r)}x_k(r)^{k-1}}{k!}rx_k'(r) \\
&= -\frac{(rp_{k-2})^2p_{k-1}rx_k'(r)}{k},
\end{aligned}$$

as claimed. With this simplification in hand, we can write Equation (A.12) as

$$\begin{aligned}
&rx_k'(r)\left[p_{k-1}-p_k-\frac{(rp_{k-2})^2p_{k-1}}{k}\right]-p_k \\
&=e^{-x_k(r)}rx_k'(r)\left[\frac{x_k(r)^{k-1}}{(k-1)!}-\frac{x_k(r)^k}{k!}-(rp_{k-2})^2\frac{x_k(r)^{k-1}}{k!}\right]-\frac{e^{-x_k(r)}x_k(r)^k}{k!} \\
&=\frac{e^{-x_k(r)}x_k(r)^{k-1}}{k!}\left[krx_k'(r)-rx_k'(r)x_k(r)-(rp_{k-2})^2-x_k(r)\right].
\end{aligned}$$

Again we substitute eq. (A.10) into the last $x_k(r)$ term, which gives

$$\frac{e^{-x_k(r)}x_k(r)^{k-1}rx_k'(r)}{k!}\left[k-x_k(r)-(rp_{k-2})^2-(1-rp_{k-2})\right].$$

This expression has the same sign as

$$k-x_k(r)-(rp_{k-2})^2-(1-rp_{k-2})=k-1-x_k(r)+rp_{k-2}(1-rp_{k-2}).$$

Hence we have shown that $S_k''(r)<0$ if and only if $k-1-x_k(r)+rp_{k-2}(1-rp_{k-2})<0$.

Step 3: Putting things together. Using the bound $x(1-x)\le\frac{1}{4}$, it suffices to prove

$$k-1-x_k(r)+\tfrac{1}{4}<0 \quad \text{that is,} \quad x_k(r)>k-\tfrac{3}{4}.$$

Lemma A.4 shows that this inequality holds for all $k \geq 4$. The case where $k = 3$ requires additional care and we provide the details in Appendix B.4. This shows that $S''_k(r) < 0$ for all $k \geq 3$, completing the proof. $\square$

A.4.3 Proof: Participation is decreasing and concave in k (part (ii))

Fix $k \geq 3$ and $r > c_{k+1}$. For notational simplicity, write $x_k \equiv x_k(r)$ and $S_k \equiv S_k(r) = \psi_k(x_k)$. We first show $S_{k+1} < S_k$ (monotonicity), then prove the discrete concavity:

$$S_{k+1} - 2S_k + S_{k-1} < 0.$$

Monotonicity in k. Since $r > c_{k+1}$ we have $x_{k+1} > 0$. Hence by Lemma A.1, $x_k > x_{k+1} > 0$, and by (A.3) we have $\psi_{k+1} = \psi_k - p_k$. So,

$$S_{k+1} = \psi_{k+1}(x_{k+1}) = \psi_k(x_{k+1}) - p_k(x_{k+1}) < \psi_k(x_{k+1}) < \psi_k(x_k) = S_k,$$

where the final inequality comes from the strict monotonicity of ψ_k ($\psi'_k = p_{k-1} > 0$ for all $x > 0$).

Discrete concavity. First, observe that by definition,

$$S_{k+1} - 2S_k + S_{k-1} = \psi_{k+1}(x_{k+1}) - 2\psi_k(x_k) + \psi_{k-1}(x_{k-1}).$$

Consider the expression

$$\psi_{k+1}(x_{k+1}) - 2\psi_k(x_{k+1}) + \psi_{k-1}(x_{k+1}) = S_{k+1} - 2S_k + S_{k-1} + G_k(x),$$

where

$$G_k(x) \equiv 2\psi_k(x_k) - \psi_{k-1}(x_{k-1}) - 2\psi_k(x_{k+1}) + \psi_{k-1}(x_{k+1}).$$

By (A.3) and (A.4), we have

$$\psi_{k+1}(x_{k+1}) - 2\psi_k(x_{k+1}) + \psi_{k-1}(x_{k+1}) = p_{k-1}(x_{k+1})\Big(1 - \frac{x_{k+1}}{k}\Big) \;<\; 0,$$

where the inequality uses Lemma A.4 ($x_{k+1} > k$). So it suffices to prove that $G_k(x) \geq 0$, that is,

$$\psi_{k-1}(x_{k-1}) - 2\psi_k(x_k) \leq \psi_{k-1}(x_{k+1}) - 2\psi_k(x_{k+1}). \tag{A.13}$$

To this end, it is convenient to define the differences

$$\Delta_- \equiv x_{k-1} - x_k > 0, \qquad \Delta_+ \equiv x_k - x_{k+1} > 0.$$

From the fixed–point identities $x_{k-1} = r\psi_{k-2}(x_{k-1})$, $x_k = r\psi_{k-1}(x_k)$ and $x_{k+1} = r\psi_k(x_{k+1})$, together with (A.3) and (A.2), we obtain the integral identities

$$\Delta_- = r\big(\psi_{k-2}(x_{k-1}) - \psi_{k-1}(x_k)\big) = r\left(\int_{x_k}^{x_{k-1}} p_{k-2}(t)\,dt + p_{k-2}(x_{k-1})\right), \tag{A.14}$$

$$\Delta_+ = r\big(\psi_{k-1}(x_k) - \psi_k(x_{k+1})\big) = r\left(\int_{x_{k+1}}^{x_k} p_{k-2}(t)\,dt + p_{k-1}(x_{k+1})\right). \tag{A.15}$$

By Lemma A.4, $x_{k+1} > k + \frac{1}{4}$, so in particular $x_{k+1} > k$. By Lemma A.3, p_{k-2} is decreasing on $[k, \infty)$; hence p_{k-2} is decreasing on $[x_{k+1}, x_{k-1}]$. Therefore

$$\int_{x_{k+1}}^{x_k} p_{k-2}(t)\,dt \geq \Delta_+\, p_{k-2}(x_k), \tag{A.16}$$

$$\int_{x_k}^{x_{k-1}} p_{k-2}(t)\,dt \leq \Delta_-\, p_{k-2}(x_k). \tag{A.17}$$

Plugging (A.16) into (A.15) and (A.17) into (A.14) gives

$$\Delta_+ \geq r\big(\Delta_+\, p_{k-2}(x_k) + p_{k-1}(x_{k+1})\big),$$
$$\Delta_- \leq r\big(\Delta_-\, p_{k-2}(x_k) + p_{k-2}(x_{k-1})\big).$$

Using the stability inequality $rp_{k-2}(x_k) < 1$ (Lemma A.2), we may divide by $1 - rp_{k-2}(x_k) > 0$ to obtain

$$\Delta_+\big(1 - rp_{k-2}(x_k)\big) \geq rp_{k-1}(x_{k+1}), \qquad \Delta_-\big(1 - rp_{k-2}(x_k)\big) \leq rp_{k-2}(x_{k-1}). \tag{A.18}$$

Dividing the two inequalities in (A.18) yields a bound on the difference ratio:

$$\frac{\Delta_-}{\Delta_+} \;\leq\; \frac{p_{k-2}(x_{k-1})}{p_{k-1}(x_{k+1})} = \frac{k-1}{x_{k-1}} \cdot \frac{p_{k-1}(x_{k-1})}{p_{k-1}(x_{k+1})} \;\leq\; \frac{k-1}{x_{k-1}} \;\leq\; \frac{k-1}{x_{k+1}} \;<\; 1, \tag{A.19}$$

where we used (A.4) and the fact that p_{k-1} decreases on $[k, \infty)$ (Lemma A.3) together with $x_{k-1} \geq x_{k+1} > k$ (Lemma A.1).

Next, define $f(x) \equiv \psi_{k-1}(x) - 2\,\psi_k(x)$. From (A.2) and (A.4),

$$f'(t) = p_{k-2}(t) - 2p_{k-1}(t) = p_{k-2}(t)\Big(1 - \frac{2t}{k-1}\Big).$$

With $u \equiv x_{k+1}/(k-1) > 1$, define $\beta \equiv \frac{2x_{k+1}}{k-1} - 1 = 2u - 1 > 0$. Then, for $t \geq x_{k+1}$,

$$f'(t) \leq -\,\beta\, p_{k-2}(t).$$

Since p_{k-2} is decreasing on $[x_{k+1}, x_{k-1}]$, we may apply Lemma A.5 which gives the bound

$$\int_{x_k}^{x_{k-1}} p_{k-2} \;\leq\; \frac{\Delta_-}{\Delta_+} \int_{x_{k+1}}^{x_k} p_{k-2}. \tag{A.20}$$

Using (A.19) and (A.20), we obtain

$$\begin{aligned}
&\big(\psi_{k-1}(x_{k-1}) - 2\psi_k(x_k)\big) - \big(\psi_{k-1}(x_{k+1}) - 2\psi_k(x_{k+1})\big) \\
&\quad = \int_{x_{k+1}}^{x_{k-1}} p_{k-2}(t)\,dt - 2\int_{x_{k+1}}^{x_k} p_{k-1}(t)\,dt \\
&\quad = \int_{x_{k+1}}^{x_k} \big(p_{k-2} - 2p_{k-1}\big)(t)\,dt \;+\; \int_{x_k}^{x_{k-1}} p_{k-2}(t)\,dt \\
&\quad \leq -\beta \int_{x_{k+1}}^{x_k} p_{k-2}(t)\,dt \;+\; \frac{\Delta_-}{\Delta_+}\int_{x_{k+1}}^{x_k} p_{k-2}(t)\,dt \\
&\quad \leq \Big(-\beta + \frac{k-1}{x_{k+1}}\Big)\int_{x_{k+1}}^{x_k} p_{k-2}(t)\,dt \\
&\quad = -\frac{(2u+1)(u-1)}{u}\int_{x_{k+1}}^{x_k} p_{k-2}(t)\,dt \;\leq\; 0.
\end{aligned}$$

Therefore we have established (A.13):

$$\psi_{k-1}(x_{k-1}) - 2\,\psi_k(x_k) \;\leq\; \psi_{k-1}(x_{k+1}) - 2\,\psi_k(x_{k+1}),$$

which completes the proof. □

A.5 Proof of Proposition 2

Suppose that $c_k < \alpha$. Observe that $r < c_k$ cannot be optimal, since $c_k < \alpha$ implies that profit is strictly increasing for all $r \in [0, c_k]$. Hence either $r^* = c_k$ is optimal, or there is some $r^* > c_k$ which is optimal. By construction, this turns on whether $\beta \lesseqgtr \overline{\beta}$. It remains to produce a bound on r^* in the latter case.

If $r^* > c_k$, then it must be that:

$$\pi(r^*) = \alpha r^* - \beta\psi_k(\rho r^*) - \frac{1}{2}(r^*)^2 > \alpha c_k - \frac{1}{2}c_k^2 = \pi(c_k).$$

By Theorem 2, the size of the giant k-core at ρr^* is greater than its size at emergence. Using ρ_c as defined in Proposition 1 this implies $\psi_k(\rho r^*) > \psi_k(x_k^\star)$, where $x_k^\star = c_k\rho(c_k)$ and so

$$\alpha r^* - \beta\psi_k(x_k^\star) - \frac{1}{2}(r^*)^2 > \alpha c_k - \frac{1}{2}c_k^2.$$

Suppose $r^* = c_k + \epsilon$. We will derive a nonzero lower-bound for ϵ. The inequality above becomes

$$\alpha(c_k + \epsilon) - \beta\psi_k(x_k^\star) - \frac{1}{2}(c_k^2 + 2c_k\epsilon + \epsilon^2) > \alpha c_k - \frac{1}{2}c_k^2.$$

$$\iff \frac{1}{2}\epsilon^2 - (\alpha - c_k)\epsilon + \beta\psi_k(x_k^\star) < 0.$$

Completing the square gives us the lower bound

$$\epsilon > 2\left(\alpha - c_k\right) - \sqrt{4\left(\alpha - c_k\right)^2 - 2\beta\psi_k(x_k^\star)},$$

which is strictly positive because $c_k < \alpha$ is a necessary condition for $r^* > c_k$ to be a maximum. Noting that this bound has the form $x - \sqrt{x^2 - a}$, we can use the difference of two squares to rewrite this as:

$$\frac{(x - \sqrt{x^2 + a})(x + \sqrt{x^2 + a})}{x + \sqrt{x^2 + a}} = \frac{a}{x + \sqrt{x^2 + a}} > \frac{a}{x},$$

from which we conclude that

$$\epsilon > \frac{\beta\psi_k(x_k^\star)}{2(\alpha - c_k)} = \frac{1}{2}\frac{\beta\psi_k(x_k^\star)}{a - c_k} > 0.$$

Putting $d_k = \frac{1}{2}\frac{\beta\psi_k(x_k^\star)}{a - c_k}$ gives the proposition.[28] □

A.6 Proof of Proposition 3

Fix $\alpha > c_k$ and take $\beta < \overline{\beta}$. Proposition 2 implies the principal's optimum satisfies $r^* > c_k$. Let

$$b(r) = \overline{a}'(r) = \psi_k'\big(r\rho(r)\big)\left[\rho(r) + r\rho'(r)\right].$$

Note that $b(r) > 0$ by Theorem 2. The principal maximizes

$$\pi(r, \beta) = \alpha r - \beta\overline{a}(r) - \frac{1}{2}r^2$$

and, because $r^* > c_k$, the optimum is interior and characterized by the first–order condition

$$F(r^*, \beta) = 0 \iff \alpha - \beta b(r^*) - r^* = 0, \tag{A.21}$$

where we define $F(r, \beta) \equiv \frac{\partial \pi}{\partial r}$.

[28] Additionally, it is fairly straightforward to use this same method to obtain the upper bound $\epsilon < 4\left(\alpha - c_k\right)$, although we did not find this bound to be particularly useful.

Step 1: sign of $dr^*/d\beta$. Totally differentiating (A.21) with respect to β yields

$$F_r(r^*, \beta)\,\frac{dr^*}{d\beta} + F_\beta(r^*, \beta) = 0 \quad \Longrightarrow \quad \frac{dr^*}{d\beta} = -\,\frac{F_\beta(r^*, \beta)}{F_r(r^*, \beta)},$$

where $F_\beta(r^*, \beta) = -b(r^*) < 0$. For the denominator, the second order condition for a maximum requires that $F_r(r^*, \beta) \leq 0$. Hence both the numerator and denominator are negative, so

$$\frac{dr^*}{d\beta} \;<\; 0. \tag{A.22}$$

Step 2: sign of $d\overline{a}^*/d\beta$. Using the chain rule,

$$\frac{da^*}{d\beta} \;=\; \overline{a}'(r^*)\,\frac{dr^*}{d\beta} \;=\; b(r^*)\,\frac{dr^*}{d\beta}.$$

Because $b(r^*) > 0$ and (A.22) gives $dr^*/d\beta < 0$, we obtain

$$\frac{d\overline{a}^*}{d\beta} < 0.$$

Finally consider the case where $\beta > \overline{\beta}$. By Proposition 2, $r^* = c_k$ and hence $a^* = 0$. By continuity of π, small changes in β leave the optimum at c_k, so $da^*/d\beta = 0$.

Thus equilibrium participation is unresponsive to β when the principal chooses the cut-off ($\beta > \overline{\beta}$) and strictly decreasing in β when she tolerates participation ($\beta < \overline{\beta}$). An identical argument, replacing β with α, yields the analogous threshold $\overline{\alpha}$. □

A.7 Proof of Proposition 4

If the principal's intervention has effectiveness σ, it acts as site percolation with retention probability $1 - \sigma$. By (Van der Hofstad, 2023, Thm 3.7) the surviving network has distribution $G(n\sigma, \frac{(1-\sigma)r}{n})$ and hence has expected degree $r' = r(1 - \sigma)$.

Part (i): By Theorem 1 (i), the equilibrium participation is positive if and only if $r' > c_k$, i.e. if and only if $\sigma < 1 - \frac{c_k}{r}$.

Part (ii): If $r' > c_k$, then by Theorem 1 (ii), among the $(1-\sigma)$ *surviving vertices*, the the fraction in the k-core is $\overline{a}^*(r') = \psi_k(r'\rho^*)$, where ρ^* is the largest solution in $[0, 1]$ to $\rho^* = \psi_{k-1}(r'\rho^*)$.

To determine the overall prevalence, consider choosing an agent uniformly at random from the original n. With probability σ that agent was converted to a good apple, and plays $a = 0$ with certainty. With probability $(1 - \sigma)$ that agent was not persuaded, in which case his probability of participating is just the probability that he is in the k-core among the surviving vertices: $\overline{a}^*(r')$. Hence by the law of total probability,

$$\overline{a}_\sigma^{\text{seed}}(r) = (1-\sigma)\overline{a}^*((1-\sigma)r)$$

which completes the proof. □

A.8 Proof of Proposition 5

Throughout this proof we continue to write $S_k(\cdot)$ for the size of the giant k-core as a function of the Erdős–Rényi mean–degree parameter (cf. Appendix A.1). Under seeding with effectiveness $\sigma \in [0,1)$, the principal chooses $r \geq 0$ to maximize

$$\pi(r;\sigma) \;=\; \alpha r - \tfrac{1}{2}r^2 - \beta(1-\sigma)\, S_k\big((1-\sigma)r\big),$$

and the equilibrium participation among the original population is

$$a^*(\sigma) \;=\; (1-\sigma)\, S_k\big(x^*(\sigma)\big), \qquad x^*(\sigma) \equiv (1-\sigma)\, r^*(\sigma).$$

Write $\psi_m'(\cdot) = p_{m-1}(\cdot)$ and recall the identities listed in Appendix A.1; in particular, $\psi_{k-1}(x_k) = x_k/r$ at the relevant fixed point (A.1) and $p_j'(\lambda) = p_{j-1}(\lambda) - p_j(\lambda)$.

Step 1: FOCs in r and in effective connectivity x. Let $x = (1-\sigma)r$. The objective can be written as

$$\Pi(x;\sigma) \;=\; \frac{\alpha}{1-\sigma}x - \frac{x^2}{2(1-\sigma)^2} - \beta(1-\sigma)S_k(x).$$

An interior optimum $x^*(\sigma) > c_k$ satisfies either of the equivalent FOCs

$$\alpha - r^* - \beta(1-\sigma)^2 S_k'\big(x^*\big) = 0, \tag{A.23}$$

$$\alpha(1-\sigma) - x^* - \beta(1-\sigma)^3 S_k'\big(x^*\big) = 0. \tag{A.24}$$

The SOC is

$$\Pi_{xx}(x;\sigma) = -\frac{1}{(1-\sigma)^2} - \beta(1-\sigma)S_k''(x) \;<\; 0 \iff D(\sigma) \;\equiv\; 1 + \beta(1-\sigma)^3 S_k''\big(x^*\big) \;>\; 0. \tag{A.25}$$

Step 2: The optimal r^* strictly falls with σ in the interior. Totally differentiating (A.23) with respect to σ and using $x^* = (1-\sigma)r^*$ and $dx^*/d\sigma = -r^* + (1-\sigma)\, dr^*/d\sigma$ yields

$$\frac{dr^*}{d\sigma} \;=\; \frac{\beta(1-\sigma)\,\Theta_k\big(x^*\big)}{D(\sigma)}, \qquad \Theta_k(x) \;\equiv\; 2S_k'(x) + xS_k''(x). \tag{A.26}$$

By the SOC, $D(\sigma) > 0$. Hence the sign of $dr^*/d\sigma$ is the sign of Θ_k.

Step 3: Equilibrium participation strictly falls with σ in the interior. Differentiate $a^*(\sigma) = (1-\sigma)S_k\big(x^*(\sigma)\big)$:

$$\frac{da^*}{d\sigma} = -\,S_k\big(x^*\big) + (1-\sigma)S_k'\big(x^*\big)\left[-r^* + (1-\sigma)\frac{dr^*}{d\sigma}\right].$$

Substitute (A.26) and use the FOCs (A.23)–(A.24) to eliminate r^* and α (multiply (A.24) by $S_k'(x^*)$ and subtract $(1-\sigma)S_k'(x^*)$ times (A.23)). This gives the clean decomposition

$$\frac{da^*}{d\sigma} \;=\; -\underbrace{\big(S_k(x^*) + x^* S_k'(x^*)\big)}_{>0} \;+\; \frac{\beta(1-\sigma)^3}{D(\sigma)}\, S_k'(x^*)\, \underbrace{\big(2S_k'(x^*) + x^* S_k''(x^*)\big)}_{=\Theta_k(x^*)}. \tag{A.27}$$

Because $S_k \geq 0$, $S_k' > 0$ on (c_k, ∞) (monotonicity in r; see Theorem 2), and $D(\sigma) > 0$, it follows from (A.27) that if $\Theta_k(x^*) < 0$ then $\frac{da^*}{d\sigma} < 0$.

Step 4: A compact sign identity for Θ_k and its sign. Write $t = t(x) = x\,\rho$ with $\rho = \psi_{k-1}(t)$ (so t is the largest fixed point corresponding to the giant k-core when the mean degree is x). A routine implicit-differentiation calculation using $\psi_{k-1}' = p_{k-2}$ and $p_j' = p_{j-1} - p_j$ yields the exact identity

$$\Theta_k(x) \;=\; \frac{\rho\, p_{k-1}(t)}{x\,\big(1 - x\,p_{k-2}(t)\big)^3}\left[(k{+}1-t) \;-\; 3\,\frac{x}{\rho}\,p_{k-2}(t)\right]. \tag{A.28}$$

At the relevant (largest) fixed point we have $1 - x\,p_{k-2}(t) > 0$ by Lemma A.2 (applied with mean degree x), so the prefactor in (A.28) is strictly positive. Hence the sign of $\Theta_k(x)$ is the sign of the bracket.

Large k. For any interior $x > c_k$, the associated fixed point satisfies $t \geq x_k^\star$, where $x_k^\star$ is the unique minimizer of $x/\psi_{k-1}(x)$ (standard fixed–point geometry; cf. the proof of Theorem 2). The argument used in Lemma A.4 to show $x_k^\star > k - \frac{3}{4}$ can be tightened (with the same telescoping ratio bound) to yield[29]

$$x_k^\star > k+1 \qquad \text{for all } k \geq 12,$$

Consequently $t \geq k+1$ at every interior $x > c_k$ when $k \geq 12$, and the bracket in (A.28) is strictly negative (the first term is ≤ 0 while the second is > 0). Thus $\Theta_k(x) < 0$ on the entire interior region for all $k \geq 12$.

Small k. For the finitely many cases $k \in \{3, \ldots, 11\}$ one can verify directly (using (A.28) together with Lemma A.2) that the bracket is negative at the interior fixed point $t = t(x)$ for all

[29]This strengthening follows by running the same computation as in Lemma A.4 with $\lambda = k+1$ (rather than $k - \frac{3}{4}$) and observing that the resulting polynomial inequality is positive for all integers $k \geq 12$. We omit the repetitive algebra.

$x > c_k$. This yields $\Theta_k(x) < 0$ on the interior also for these k.

Combining the two cases, $\Theta_k(x) < 0$ whenever $x > c_k$.

Step 5: Corners. If $x^*(\sigma) = c_k$ (the "hold–at–the–cutoff" corner), then $S_k(c_k) = 0$ and hence $a^*(\sigma) = 0$ with $\frac{da^*}{d\sigma} = 0$. If $(1-\sigma)\alpha \leq c_k$ (the "always safe" region), then $r^*(\sigma) = \alpha$, $x^*(\sigma) = (1-\sigma)\alpha \leq c_k$, so again $a^*(\sigma) = 0$ and $\frac{da^*}{d\sigma} = 0$.

Putting the pieces together: by (A.26) and (A.27), $\Theta_k(x^*) < 0$ implies $dr^*/d\sigma < 0$ and $da^*/d\sigma < 0$ at every interior optimum; in the two corner regions the derivative of a^* is 0. Therefore, whenever equilibrium participation is nonzero (equivalently, $x^*(\sigma) > c_k$), it is strictly decreasing in σ. □

B Online Appendix

B.1 Micro-founding the largest equilibrium

In Section 4 we assumed that agents play the largest equilibrium in the second stage (i.e. the one where the most agents take the action $a_i = 1$). Focusing on the largest equilibrium is a natural benchmark, because (i) it provides a worst-case scenario for the principal, and (ii) the other extreme—the smallest equilibrium—is trivial. The smallest equilibrium always involves no participation.

Here, we also show two different ways of micro-founding our focus on the largest equilibrium. In the first, agents boundely-rational ("unsophisticated") and play myopic best responses. In the second, agents are fully rational ("sophisticated") and reach equilibrium using only local information and a short communication phase.

Unsophisticated Agents: Myopic Best Response Dynamics

Set-up. Suppose that the game is now played over $n+1$ periods. At $t=1$, the principal chooses $r \geq 0$, and Nature realizes the graph G (as in our baseline model). Additionally, at $t=1$, all agents participate *non-strategically* (i.e. $a_{i,t=1} = 1$ for all i). In each subsequent period $t \geq 2$, all agents best respond myopically to actions in the preceding period $t-1$. Agent i's action in period $t \geq 2$, $a_{i,t}$ in the game $\Gamma^{(n)}$ is thus determined by the best response

$$BR_{i,t}^{(n)}(M_{i,t-1}) = \begin{cases} 1, & \text{if } M_{i,t-1} \geq k \\ 0, & \text{if } M_{i,t-1} < k, \end{cases} \tag{B.1}$$

where $M_{i,t-1} = \sum_{j \neq i} G_{ij} a_{j,t-1}$. For convenience, we use $a_i \equiv a_{i,n+1}$ to denote i's action in the final period of the game. And we assume that the principal cares about participation at the end of the game—not during it.

The result. Exactly as in our baseline model, the unique limit of the above dynamic process is that each agent participates if and only if they belong to the k-core of the graph G. In other words,Remark 1 holds unchanged.

Discussion. In addition to myopia, this set-up assumes that all agents start off by participating. From a technical standpoint, it guarantees convergence of the best response dynamics to the largest equilibrium of the static game. The logic of how it works is very similar to that laid out after Remark 1—the only difference is that agents are updating myopically, rather than doing analogous reasoning in their head.

Economically, this set-up imposes $a = 1$ as the pre-existing default option. So this myopic updating view is best fits settings where the principal wants to eradicate or reduce pre-existing behaviors. This is very natural when considering adoption of new technologies or new social norms—the new versions simply did not exist before the start of the game, so everyone must therefore have been using the old technology or old norm. Note that in this interpretation, our terminology of 'participate' or 'abstain' becomes less clean—'participate' would mean to use the *old* technology/norm, and 'abstain' would mean to use the *new* one.

Relatedly, assuming myopic best responses effectively rules out coordination between agents. We contend that this is another reason why focusing on the largest equilibrium provides a useful benchmark.

Sophisticated agents: local communication

Suppose that agents start in a "communication phase" that lasts $n + 1$ periods, followed by an "action phase" at $t = n + 2$. At $t = 1$, the principal chooses $r \geq 0$, and Nature realizes the graph G. In each period $t \geq 2$ agents observe only (i) how many neighbors they have and (ii) what each of those neighbors announced in the previous period.

Communication phase ($2 \leq t \leq n + 1$). During the communication phase, every agent i announces a *cheap talk* message $m_{i,t} \in \{P, A\}$, interpreted as "I Plan to participate" or "I will Abstain". Suppose agents adopt the rule:

$$m_{i,t} = \begin{cases} \text{A} & \text{if strictly fewer than } k \text{ neighbours sent P at } t-1, \\ \text{P} & \text{otherwise.} \end{cases}$$

We assume all agents start with $m_{i,1} \equiv \text{P}$.

Action phase ($t = n + 2$). Consider the strategy s_i given by: play $a_i = 0$ if i ever sent A during the communication phase, and play $a_i = 1$ if i sent P in all n rounds of the communication phase.

The result. The message rule implements the usual k-core peeling process: after (at most) n periods, no further A's are sent and the only agents who still announce P are the vertices in the k-core of G.

Given the strategy s_i for the action phase, any agent who still announces P in the final communication round has at least k neighbors who also announced P. These agents will therefore choose $a = 1$ in the action phase. Hence playing $a_i = 1$ is a best response for every agent who announces P at time $n+1$, that is, for every agent in the k-core. Similarly, any agent who sent A expects less than k of their neighbors to play $a = 1$, and therefore their best response is to play $a_i = 0$. Thus $s = (s_i)_{i \in N}$ is a Nash equilibrium of the action phase.

In the communication phase, no agent can profit (in any period) by unilaterally deviating from the message rule. Sending P when the rule prescribes A cannot push the number of remaining P-neighbors back to k, so it cannot raise the deviator's final payoff. Sending A when the rule prescribes P only removes an agent from the set of potential participants and therefore strictly lowers his payoff if he was in the k-core. Therefore the prescribed message profile $m = (m_{i,t})_{i \in N, 1 \le t \le n+1}$ is itself incentive-compatible.

Consequently, (m, s) is a Nash equilibrium in every subgame and therefore a subgame-perfect equilibrium of the entire game. The agents who play $a_i = 1$ in the terminal period are exactly those in the k-core of G.

Discussion. Agents here are fully rational and use the communication phase to perform the same k-round elimination that the "unsophisticated" agents achieve through myopic play. Crucially, each agent needs only *local* information: his own degree and the most recent messages of his neighbors. The procedure demonstrates that the k-core offers a natural characterization of equilibrium even when agents are fully sophisticated, as in our baseline model (see Remark 1).

B.2 The Discontinuous Emergence of the k-core: Intuition

Suppose we choose a participating vertex at random in the graph and follow it to one of its neighbors, say i.

Consider the $k = 2$ case. For i to be part of a group which sustains the behavior, we only need to find one other participating neighbor. This is a relatively weak requirement— it allows for a small cluster of agents to be self-sustaining. If the hypothetical fraction of participants in the network is very small (say, ρ), an agent's chance of finding at least one participating neighbor is also small (roughly proportional to ρ). This allows for a stable outcome where a tiny fraction of agents can mutually sustain each other, and this fraction grows continuously as the network becomes denser.

Now, consider the $k = 3$ case. For i to be part of a group which sustains the behavior, we need to find two other participating neighbors. Because the probability of finding multiple participating neighbor compounds, this is a much stronger requirement. If the fraction of participants ρ is very small, the probability of any given agent finding two participating neighbors is drastically smaller (roughly proportional to ρ^2). A tiny self-sustaining group is therefore impossible; the demanding requirement for participation cannot be met if the pool of potential supporters is itself vanishingly small. The system cannot be "born small." For a k-core with $k \ge 3$ to exist at all, the network must already be dense enough to support a large concentration of participants, leading to the discontinuous jump from zero to a substantial fraction of the population.

This logic can be formalized through branching process approximations of the local network structure.

B.3 Equilibrium is Generically Unique

By Proposition 2, there are two possible equilibria: $r^* = c_k$, or $r^* > c_k + d_k$. Note that since profit is strictly decreasing for all $r > \alpha$, any maximizer of the profit function must belong to the interval $[c_k + d_k, \alpha]$.

Since π is continuous, it attains a global maximum on this compact interval. Moreover, since $\psi(\cdot)$ is analytic and $\rho(r)$ is analytic, so is π. In particular π' is also analytic (and not identically 0). An analytic function that is not identically zero has only isolated zeros, and therefore π' must have only finitely many zeros in $[c_k + d_k, \alpha]$.

Thus the set

$$\text{argmax}_{r \in [c_k + d_k, \alpha]} \pi(r)$$

is both nonempty and finite.

Now suppose that there are two distinct maximizers $r_1^* < r_2^*$. Consider what happens when we decrease β slightly. Since $r_2^* > r_1^*$, it follows from theorem 2 that $S_k(r_1^*) < S_k(r_2^*)$. Hence for any $\epsilon > 0$, this implies

$$\begin{aligned}
\alpha r_1^* - \frac{1}{2}(r_1^*)^2 - (\beta - \epsilon) S_k(r_1^*) &= \left(\alpha r_1^* - \frac{1}{2}(r_1^*)^2 - \beta S_k(r_1^*)\right) + \epsilon S_k(r_1^*) \\
&= \alpha r_2^* - \frac{1}{2}(r_2^*)^2 - \beta S_k(r_2^*) + \epsilon S_k(r_1^*) \\
&< \alpha r_2^* - \frac{1}{2}(r_2^*)^2 - \beta S_k(r_2^*) + \epsilon S_k(r_2^*),
\end{aligned}$$

and so r_2^* emerges as the unique maximizer. Since π is continuous in β, we can always perturb it by some small enough ϵ such that another "new" maximizer does not appear.

A similar argument can be made if both $r_1^* = c_k$ and $r_2^* > c_k$ are maximizers. So equilibrium is generically unique.

B.4 Concavity when $k = 3$

In appendix A.4.2 we showed that $S_k''(r)$ has the same sign as

$$x_k'(r)(p_{k-1} - p_k)(r - r^2 p_{k-2}) - p_k + 2r p_k p_{k-2} + x_k'(r) r^2 (p_k p_{k-3} - p_k p_{k-2}).$$

Substituting $k = 3$ and using the identities:

$$\begin{aligned}
x_3(r) &= x_3'(r)(r - r^2 p_1) \\
p_2 - p_3 &= \frac{e^{-x} x^2}{6}(3 - x) \\
p_0 - p_1 &= e^{-x}(1 - x),
\end{aligned}$$

we see that $S_3''(r)$ has the same sign as

$$\frac{e^{-x_3(r)}x_3(r)^2}{6}\left[(3-x_3(r))-1+2rp_1+\frac{rx_3(r)e^{-x_3(r)}}{1-rp_1}\right].$$

Let $t=1-rp_1=1-rx_3(r)e^{-x_3(r)}$. Then since $\frac{(1-t)(1-x)}{t}=\frac{1-x}{t}-(1-x)$, the terms in the square brackets above can be rewritten as

$$(3-x_3(r))+(-1-2(1-t))+\frac{1-x_3(r)}{t}-(1-x_3(r))=3-\left(2t+\frac{x_3(r)-1}{t}\right),$$

so it suffices to prove that

$$\left(2t+\frac{x_3(r)-1}{t}\right)\geq 3$$

for all $r\geq c_3$. We prove this in cases.

Case 2: $x_3(r)\in[x_3(c_3),\,\frac{17}{8}]$**.** Numerical methods show that $x_3(c_3)\approx 1.79$. So it suffices to prove the claim for $x_3(r)\in[\frac{7}{4},\,\frac{17}{8}]$.

Since, in equilibrium, $r=\frac{x_3(r)}{\psi_2(x_3(r))}$, we have that $t=t(x)=1-r(x)p_1(x)$ can be written as

$$t(x)=\frac{1-e^{-x}(1+x+x^2)}{\psi_2(x)}=\frac{1-e^{-x}(1+x+x^2)}{1-e^{-x}(1+x)}.$$

Differentiating with respect to x gives

$$t'(x)=\frac{e^{-x}x\big((x-1)\psi_2(x)-[1-e^{-x}(1+x+x^2)]\big)}{\psi_2(x)^2}=\frac{e^{-x}x\big((x-2)+e^{-x}(2+x)\big)}{\psi_2(x)^2}.$$

The sign of t therefore depends on $g(x)\equiv(x-2)+e^{-x}(2+x)$. Since $g(0)=0$, and $g'(x)=1-e^{-x}(1+x)>0$, it follows that t is increasing for all $x>0$. So for $x\in[\frac{7}{4},\,\frac{17}{8}]$,

$$t(x)\leq t\big(\tfrac{17}{8}\big)=\frac{1-e^{-17/8}\left(1+\frac{17}{8}+(\frac{17}{8})^2\right)}{1-e^{-17/8}\left(1+\frac{17}{8}\right)}\approx 0.140\ <\ \tfrac{1}{4}.$$

Hence

$$2t(x)+\frac{x-1}{t(x)}\geq\frac{x-1}{t(x)}\geq\frac{7/4-1}{1/4}=3,$$

which proves case 2. Hence $S_3''(r)<0$ for all $r\geq c_3$, as claimed.