Controlling Complex Contagions*

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Abstract

Governments often restrict communication to prevent people from coordinating protests and popular uprisings. But communication is also essential to economic activity, which creates a tradeoff for governments. We build a theoretical model to study this tradeoff. Our model shows: (i) how and why large protests occur suddenly, (ii) why regimes act to reduce the rate of interactions even when no protest is observed, and (iii) that homogeneity in the structure of interactions both exacerbates the size of protests and pushes the government to extreme interventions.

JEL Codes: D74, D85, Z13. *Keywords:* complex contagion, protest, collective action, repression, network design, random networks, tipping points.

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1 Introduction

In August 2019, Indian authorities cut off mobile, internet, and landline communication in the Kashmir Valley. Yet at the time it was imposed, there was no widespread protest. Although the shutdown disrupted livelihoods across the Valley, the government defended it as necessary to prevent *anticipated* civil unrest. But restrictions on communication need not be all-or-nothing. When connectivity returned to the Kashmir Valley in early 2020, it returned at 2G speeds—fast enough to text but not fast enough for video. More recently, for three weeks in March 2026, the Russian government shut down mobile internet but *not* landline internet in Moscow.[1] Yet indiscriminate shutdowns are alarmingly common: the digital-rights organization Access Now has recorded 2,102 internet shutdowns across 100 countries between 2016 and 2025.

These choices illustrate a fundamental tradeoff: governments benefit from the economic value created by communication among their citizens,[2] but communication also facilitates the contagious spread of harmful collective action. But why do some governments opt for extreme policies—shutting down communication entirely—while others impose only moderate restrictions?

To understand this, we develop a theory of collective action and network design in which a government values interactions between citizens but is harmed by the collective action that those same interactions can sustain. Consistent with experimental findings that protest participation exhibits local complementarities (Bursztyn et al., 2021), citizens in our model participate only if enough of their neighbors in a network do too (a feature known as *complex contagion* in the networks literature (Centola and Macy, 2007)).

Concretely, we study a two-stage game between a principal (the government) and a large population of agents (citizens) who interact in a random network. The principal moves first: she controls only a single lever, the aggregate *interaction rate* r, which determines the average number of links agents have. She cannot shape the microstructure of the network. Additionally, agents have heterogeneous propensities

[1] See Human Rights Watch (2026) for details. The government explicitly justified the restrictions on the grounds of security.

[2] Communication technology supports the creation of economic value. For example, Jensen (2007) finds that the introduction of mobile phones in South Indian fisheries increased both producer and consumer welfare.

to form links, summarized by a CDF F (the *interaction profile*). The number of links an individual agent forms depends on the overall interaction rate scaled by their personal propensity to form links. Links form organically through chance interactions, not by design. With the network in place, each agent then chooses whether to participate in a collective action, and prefers to participate if and only if at least $k \geq 3$ (the *participation threshold*) of his neighbors in the network do as well. The principal values interaction directly, but bears a cost when agents participate in the collective action.

Our first set of results characterizes the relationship between the aggregate interaction rate (i.e. network connectivity) and equilibrium participation. We find that in the largest equilibrium—which is the worst case for a government concerned about protest[3] —there is a critical cutoff of the interaction rate where participation in the collective action jumps discontinuously from zero to a positive share of agents.[4] We write $c_k(F)$ for the critical cutoff and $J_k(F)$ for the size of the jump.

Importantly, the size of the jump is shaped by heterogeneity in the interaction profile (Figure 1, left panel). It is largest when the interaction profile is homogeneous. Moreover, this jump is economically meaningful—more than a quarter of the population when the participation threshold is three neighbors.[5] The jump is strictly decreasing in heterogeneity under the *star* order. Informally, one profile is more concentrated than another in the star order when there is more inequality: relatively inactive agents have smaller propensities to interact and relatively active agents have larger ones. So when interactions are concentrated in a small core of agents, participation is confined to that core and the jump can be made arbitrarily small.[6] As we

[3] Governments often act against the *capacity* for protest—precisely what is captured by the largest equilibrium—rather than protest itself. For example, King et al. (2013) show that the Chinese state censors social media posts with collective-action potential even when those posts *support* the government—it is coordination, not hostility, that governments act against. Communication networks create this coordination potential—the expansion of social media increased protest participation in Russia (Enikolopov et al., 2020).

[4] These results are obtained by analyzing the k-core of the random network—the largest subgraph in which every agent has at least k connections to other members. The k-core is known to characterize equilibrium behavior in threshold games on finite, deterministic networks (Gagnon and Goyal, 2017; Langtry et al., 2024); we work with its large-network analogue.

[5] In a controlled experiment, Centola et al. (2018) show that a committed minority of about 25% can flip a prevailing social convention. We view this as qualitative evidence for discontinuities in complex contagions, consistent with our model.

[6] This connects to a long-standing theme in the study of collective action—not just when participation tips, but who sustains it. Early work on critical mass argued that collective action is often carried by a small, intensely engaged group (Oliver et al., 1985; Marwell and Oliver, 1993);

will see, this can help explain why some governments reach for internet shutdowns while others opt for more moderate policies.

The location of the jump—the value of the critical cutoff—also depends on heterogeneity in the interaction profile, but through a different order. Holding the mean fixed, one profile lowers the critical cutoff relative to another exactly when it shifts up the distribution of *forward degrees*—the connectivity of a neighbor of a randomly chosen agent—in the sense of first-order stochastic dominance. This is called the *starshaped* order (Shaked and Shanthikumar, 2007), which, despite the similar name, is distinct from the star order that ranks the jump. More concentration in this sense lowers the cutoff, so participation becomes possible at lower levels of aggregate connectivity. The core–periphery case in Figure 1 belongs to a family ranked in both orders. Thus the two objects that shape the principal's problem—the jump and the cutoff—respond to heterogeneity in different but interpretable ways.

Our second set of results characterizes the principal's optimal choice of connectivity. The discontinuity has important implications for the government: it can never face less protest than $J_k(F)$, unless it suppresses protest completely. As such, interaction rates just above the cutoff can never be optimal—moving past the cutoff entails a large jump in the cost of collective action while delivering a comparatively small gain from the extra interactions. This creates a *missing middle* in policy: intermediate choices of the aggregate interaction rate are never optimal, and the government is pushed towards extreme measures. It is this stark, all-or-nothing logic that can rationalize responses such as internet shutdowns. But the missing middle is only as wide as the jump: as heterogeneity shrinks $J_k(F)$, the missing middle narrows, and moderate policy becomes feasible again. Our theory thus provides an answer to the question we began with: whether a government reaches for extreme versus moderate policies depends on how interaction is spread across its population.

The left panel of Figure 1 illustrates the contrast for two interaction profiles: in a homogeneous environment participation jumps by roughly 27% of the population; in a core–periphery environment with the same average connectivity, participation jumps by roughly 5%. Figure 2 shows the principal's payoff in each environment: in the homogeneous environment the large jump produces a sharp drop in payoff at the cutoff, and with it a wide missing middle; in the core–periphery case the drop is small and the missing middle largely disappears.

our model microfounds that intuition.

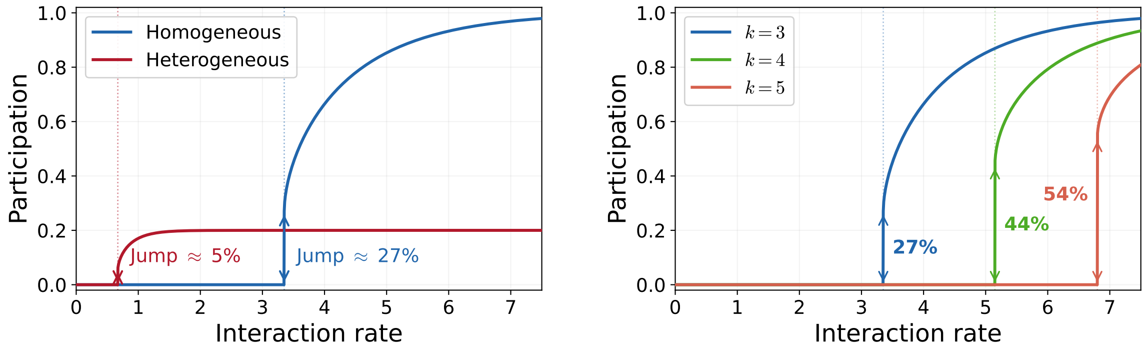

Figure 1: *Left:* Equilibrium participation as a function of network connectivity in two environments with the same average degree: a homogeneous environment (blue) and a core–periphery environment (red). The arrows indicate the size of the jump. In the homogeneous environment, participation jumps by roughly 27%; in the core–periphery environment, by roughly 5%. *Right:* Equilibrium participation in the homogeneous environment for participation thresholds $k = 3, 4, 5$. Higher thresholds require more connectivity and produce larger jumps.

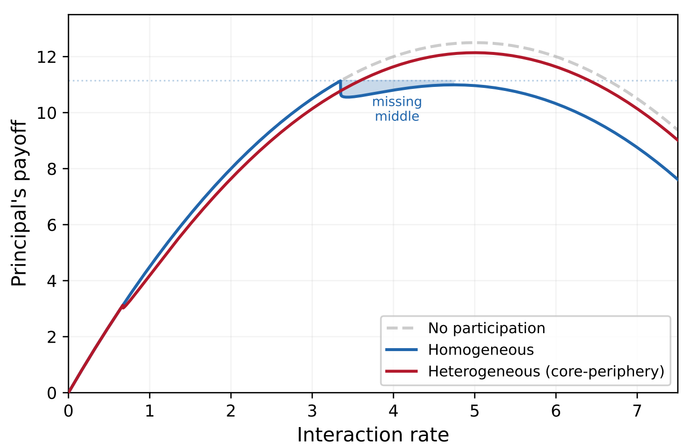

Figure 2: The principal's payoff in the homogeneous environment (blue) and core–periphery environment (red). The dashed gray curve represents an environment with no participation. The sharp drop in payoffs at the cutoff in the homogeneous case reflects the large jump in participation.

Finally, our analysis delivers results of independent interest. Although the relevant mathematical object—the k-core of a large random network—has been well studied, we provide comparative statics that, to our knowledge, were not previously known. Above the critical cutoff, participation is strictly increasing in the interaction rate r and strictly decreasing in the participation threshold k. The critical cutoff $c_k(F)$ also rises as k increases (Figure 1, right panel). Moreover, the principal's problem has a tipping point of its own: near the boundary between suppressing participation and tolerating it, small changes in her incentives flip which one she chooses.

While citizens protesting against a government is our leading example, our model captures a tension between the benefits of connectivity and the potential harms of collective action that appears in a range of settings: firms value collaboration among workers that can also sustain misconduct or unionization, and platforms are built on engagement that can also sustain misinformation. We return to these settings in the conclusion.

The rest of the paper is organized as follows. Section 2 discusses related literature. Section 3 sets out the model. Section 4 characterizes agents' equilibrium behavior and the discontinuous emergence of participation. Section 5 examines the location and size of the jump and how they depend on the interaction profile. Section 6 solves the principal's design problem and derives the missing-middle result. Section 7 concludes.

2 Related Literature

Our paper builds on and contributes to four main literatures.

First, we develop a tractable model of complex contagions on a random network, and derive sharp comparative statics for equilibrium behavior, which have been elusive in fixed-network models. There is a large literature that studies models of complex contagion, with applications to a very wide range of economic behaviors.[7] Existing work here primarily uses fixed networks (see, e.g. Gagnon and Goyal, 2017; Reich, 2026; Chandrasekhar et al., 2025), and often focuses on the diffusion of the action from some starting group of agents (Morris, 2000; Acemoglu et al., 2011).[8] More broadly, agents in our model play a game of strategic complementarities (Bulow et al., 1985; Milgrom and Roberts, 1990; Vives, 1990).

[7]These include technology adoption (Reich, 2026), protest (Chwe, 2000), pricing (Zhang, 2025), financial contagion (Rogers and Veraart, 2013; Elliott et al., 2014), persuasion (Candogan, 2022), and choices of whether to participate in formal markets (Gagnon and Goyal, 2017). Granovetter (1978) and Schelling (1978) also suggest many other applications.

[8]Another approach has been to push the network into the background, by abstracting from it altogether (Granovetter, 1978; Schelling, 1978) or imposing a "mean field" assumption (Jackson and Yariv, 2006; López-Pintado, 2006, 2008), or to assume that agents only have partial information (Galeotti et al., 2010; Leister et al., 2022).

Second, by developing a model of collective behavior on a random network, we connect to a growing literature on simple contagion—where agents need only *one* neighbor to act—on random networks (Campbell, 2013; Akbarpour et al., 2025; Campbell et al., 2024,?; Langtry, 2025). Sadler (2020) studies a general class of diffusion games on configuration model random graphs that are closely related to ours. A key distinction is that simple contagions produce continuous emergence of viral behavior: the giant component grows smoothly as connectivity increases (Centola and Macy, 2007; Centola, 2010). Complex contagions, by contrast, produce *discontinuous* transitions—participation jumps from zero to a positive fraction at the critical cutoff. Our model makes this contrast precise, and provides a tractable class of random networks on which it can be studied. Campbell et al. (2024) also study a mixed Poisson random graph in a model of entry deterrence and pricing, focusing on the role of the giant component.

Third, we contribute to a literature on discontinuous phase transitions in networks—settings where small changes in network structure induce large, discontinuous changes in outcomes. Watts (2002) provides the seminal analysis of fractional-threshold contagion on random networks, showing that cascades can fail entirely or succeed on a large scale, with little in between. Elliott et al. (2022) show how a need for multiple types of input in supply networks can produce analogous discontinuous transitions, and provide comparative statics for the fragility of such networks. In our model, the discontinuity arises directly from the nature of threshold-based complex contagion itself: agents require a fixed number of participating neighbors, and this creates a sharp tipping point in aggregate behavior. We provide a number of new comparative statics and characterize how the distribution over agents' propensities to interact shapes the size and location of the discontinuity—including a global extremality result (homogeneity maximizes the size of the discontinuity) and stochastic orders that rank the size and location of the jump.Jackson and Storms (2026) also study complex contagion on random networks using stochastic block models, but focus on identifying which groups must take the same action and on optimal seeding, rather than on comparative statics for aggregate participation.[9]

Finally, we embed our analysis within a principal's optimal network design prob-

[9] Their goal is to understand (a) which groups of agents must take the same action in any equilibrium, and (b) how to use this knowledge to best pick a set of initial adopters to maximize the spread of a new behavior. Our focus is instead on comparative statics for overall equilibrium behavior with respect to features of the network, as well as how a principal chooses overall network connectivity.

lem where the principal chooses aggregate connectivity. We show that homogeneity-versus-heterogeneity in the network structure influences whether the optimal policy is always extreme or can be interior. This connects us to the literature on network design and formation. This literature has increasingly focused on settings where a network forms and then agents play a game on the resulting network (Galeotti and Goyal, 2010; Sadler and Golub, 2021; Kinateder and Merlino, 2017, 2022). In part due to technical challenges, there is little work on network formation in a random-network setting. Elliott et al. (2022) and Langtry (2025) are exceptions, but consider endogenous formation by agents rather than control of overall connectivity by a principal. Galeotti et al. (2020) study the design of interventions in networks where a planner targets agents based on their degree. In a significantly different setting, Acemoglu et al. (2024) study platform design in a misinformation context and also find that the optimal policy can take a bang-bang form: the platform either allows wide sharing or restricts it sharply. Our analysis complements theirs by showing that this extreme logic is not universal—it depends on the heterogeneity in agents' propensities to form links.

Closest in spirit to us is Lorentzen (2014), who studies an authoritarian regime that values investigative reporting on local officials, but fears that the same reporting could let citizens coordinate a revolt if it makes the true extent of social discontent common knowledge. The tradeoff we study is similar, but modelling the contagion process explicitly allows us to answer a much richer set of questions.

3 Model

We consider a sequence of games $\{\Gamma^{(n)}\}_{n\in\mathbb{N}}$ indexed by the number of agents n. We now describe the structure of a specific game $\Gamma^{(n)}$.

Agents, Actions & Timing. There is a single principal ("she"), P, and n agents ("he"), indexed $i \in N \equiv \{1, \dots, n\}$. In the first period ($t = 1$), the principal chooses an *interaction rate* $r \geq 0$, taking an *interaction profile* F—a distribution over agents' propensities to interact, defined formally below—as given.[10] Then Nature forms a

[10]We view F as a slow-moving, difficult-to-change feature of the environment, while r can be manipulated. Exploring what happens when the principal can also shape F is an interesting direction for future work; see Section 7.

large random network among agents (which we describe shortly). In the second period ($t = 2$), each agent simultaneously chooses whether or not to take a binary action, $a_i \in \{0, 1\}$. For clarity, we will say that an agent who takes action 1 *participates*, and an agent who takes action 0 *abstains*.

Random network formation and interaction profile. The principal controls only the interaction rate r (i.e. the network connectivity). But agents differ in their propensity to interact and form links. Each agent i draws an i.i.d. non-negative interaction weight W_i from a distribution F. This captures heterogeneity in interactions: i's expected degree is $r \times W_i$.[11] The interaction weights W_i capture this variation. We impose the following condition on the interaction profile.

Assumption 1. *Weights $(W_i)_{i=1}^n$ are i.i.d. with $W_i \sim F$, satisfying $\mathbb{E}[W] = 1$, and $\mathbb{E}[W^2] < \infty$.*[12]

The assumption that $\mathbb{E}[W] = 1$ is simply a normalization. Conditional on W_i and r, each agent draws a number of stubs (their degree) $D_i \mid W_i \sim \text{Poisson}(rW_i)$ independently across i. Then conditional on the realized degree sequence $(D_i)_{i=1}^n$, Nature forms a (simple) random graph via the *configuration model*: stubs are paired uniformly at random, and we condition on the resulting graph being simple (i.e. no self-loops or multiple edges).[13] We let $G^{(n)}$ denote the adjacency matrix of a realization of this random graph: $G_{ij}^{(n)} \in \{0, 1\}$.

Preferences: principal. The principal values interactions between agents, but bears a cost when agents *participate* in the undesirable behavior. Providing the underlying interaction capacity—opportunities to engage, communication infrastructure, meeting spaces—is also costly. We capture the principal's net benefit from interactions, before accounting for participation, with a simple linear-quadratic form: $\alpha r - \frac{1}{2}r^2$, where $\alpha > 0$ is the marginal benefit of interaction and $\frac{1}{2}r^2$ a convex cost. Online Appendix OA.2 microfounds the linear benefits in r as the equilibrium value of

[11]Such heterogeneity is well-documented empirically: degree distributions in real-world networks typically have a small fraction of agents maintaining many more connections than the median (Jackson, 2008).

[12]While mild, this can be violated by sufficiently heavy-tailed profiles. We discuss this in Online Appendix OA.3.

[13]Under Assumption 1, the configuration model produces a simple graph with asymptotically positive probability, so this conditioning is without loss.

a game where agents take bilateral actions with each of their friends, and the principal benefits from these actions.[14] Let $\beta > 0$ denote the per-capita cost she bears when agents participate, and $\bar{a} = \frac{1}{n}\sum_i a_i$ denote the *participation rate*. The principal's per-capita payoff at interaction rate r is

$$\pi(r, F, \bar{a}) \;=\; \alpha\, r \;-\; \beta\, \bar{a} \;-\; \frac{1}{2}r^2. \tag{1}$$

Note that the interaction profile F only affects the principal's objective through equilibrium participation.

The two cost terms have distinct economic interpretations. *Infrastructure cost* ($\frac{1}{2}r^2$): the cost of providing interaction opportunities at rate r, regardless of how connections are distributed. This is an *extensive-margin* cost: it depends on the total volume of interaction, not how those interactions are distributed across agents. *Participation cost* ($\beta\bar{a}$): the cost the principal bears when agents participate.

Preferences: agents. Agents' actions are strategic complements: participating is more attractive to agent i when more of his neighbors participate. Specifically, i prefers to participate (choose $a_i = 1$) if and only if at least $k \geq 3$ of his neighbors also participate, where k is the *participation threshold.* Agents' preferences can be represented by a utility function $u_i(a_i, M_i)$ such that:

$$u_i(1, M_i) \geq u_i(0, M_i) \quad \iff \quad M_i \geq k, \tag{2}$$

where $M_i = \sum_{j \neq i} G_{ij}^{(n)} a_j$ is the number of participating neighbors of i.

Information. For expositional simplicity, we assume that the realization of the network is common knowledge to agents. This means that actions a_i are functions $a_i = a_i(G^{(n)})$ of the realized network $G^{(n)}$ observed by agents. In Online Appendix OA.1 we provide a microfoundation which demonstrates that our model here can be viewed as the reduced-form of a model in which agents participate in an explicit diffusion process and only observe the actions of their neighbors.

[14] But neither the linearity of the benefit nor the quadratic cost are essential: our qualitative results only require that the principal's benefit, absent participation costs, is twice continuously differentiable and strictly concave with an interior maximum.

Equilibrium. A strategy profile $\left(r^\star, a_1^\star, a_2^\star, \ldots, a_n^\star\right)$ is a *subgame perfect Nash equilibrium* of the game $\Gamma^{(n)}$ if, for any realization of the graph $G^{(n)}$, we have:

$$\text{for all } i \in N, \quad a_i^\star = 1 \iff u_i(1, M_i^\star) \geq u_i(0, M_i^\star),$$

and the principal chooses r^* such that

$$r^* \in \arg\max_{r \geq 0} \mathbb{E}_{\mathbb{P}_{r,F}}\left[\pi(r, F, \bar{a}^*)\right],$$

where $\bar{a}^* = \frac{1}{n}\sum_{i=1}^n a_i^*$ is the equilibrium level of participation in the second stage and $\mathbb{P}_{r,F}$ is the distribution over networks generated by the principal's choice of r and the interaction profile F.

This definition requires that (i) agents best-respond to other agents given the realized network, and (ii) the principal maximizes her expected payoff, anticipating both the network structure and the resulting actions. In this paper we focus on the largest (i.e. highest participation) equilibrium, because it captures the worst feasible self-sustaining outcome for the principal and is also selected by natural local best-response dynamics. Online Appendix OA.1 formalizes the latter interpretation.[15]

4 How Agents Behave

We begin with the second stage of the game, and characterize how agents' behavior depends on the interaction rate r and the interaction profile F. The second stage is a threshold game played on the realized network G. So before going further, it is important to discuss the role of the *k-core*—a notion of "cohesive" parts of a network—in characterizing equilibrium behavior.

4.1 Equilibrium and the k-core

In threshold games on networks, the largest equilibrium is determined by the network's "core structure". The k-core of a graph G is the largest induced subgraph H of G such that every agent in H has degree at least k within H (Seidman, 1983). In

[15]Agents' actions are strategic complements, so the second stage of the game on any realized network $G^{(n)}$ has a *complete lattice* of Nash equilibria under the component-wise order on $\{0,1\}^n$ (Milgrom and Roberts, 1990; Topkis, 1998). In particular, there exists a *largest* equilibrium $a^{\max}(G^{(n)})$—the one with the most participating agents.

our model the set of agents who participate in the largest equilibrium is precisely the k-core of the network G.

Remark 1. *In the largest equilibrium, agent i participates ($a_i = 1$) if and only if he belongs to the k-core of the realized network $G^{(n)}$.*[16]

We can see why this is the case by considering how agents might reason about a stable outcome. Any agent with fewer than k neighbors in the entire network knows he can never meet the participation threshold. So he will choose to abstain. Knowing this, other agents may revise what they expect their neighbors to do—for example, an agent who initially had k neighbors might now expect some of them not to participate, causing him to also abstain. Continuing in this way, we can iteratively remove agents who expect $< k$ participating neighbors until only a stable group remains. The agents left are exactly those in the k-core; each agent in it has at least k neighbors who also participate.

A key implication of Remark 1 is that characterizing equilibrium behavior in our model amounts to characterizing the fraction of agents belonging to the k-core of the random network induced by (r, F). The challenge in fixed-network models has been that it is very difficult to make strong predictions about how equilibrium behavior changes as a function of the primitives of the network. Using random networks allows us to obtain clean results in the limit of a large population. Therefore the remainder of our analysis focuses on the asymptotic properties of the model as the number of agents $n \to \infty$. As such, the equilibria we characterize are limits of subgame perfect equilibria of the finite games $\Gamma^{(n)}$.[17] We will often omit the limit in our statement of results, but any such results should be understood as applying with high probability as $n \to \infty$.

4.2 Equilibrium behavior

Our characterization of equilibrium behavior in the second stage builds on established results from random graph theory. The existence of a *giant k-core* (a k-core containing a constant fraction of agents with high probability as $n \to \infty$) is known to

[16] The direct link between the largest equilibrium and the k-core has been established in prior studies of threshold games on fixed (finite) networks (Gagnon and Goyal, 2017; Langtry et al., 2024).

[17] All references to "equilibrium" from here on should be taken to mean "limits of equilibria in the sequence of finite games".

depend on a sharp cutoff condition.[18] Above this cutoff, the size of the giant k-core is strictly increasing in network connectivity. This provides the crucial link between the interaction rate r and equilibrium participation.

To fix ideas, let $a_k(r;F) \equiv \lim_{n\to\infty} \frac{1}{n}\sum_i a_i^*$ denote the limiting fraction of participating agents—the (level of) participation—given an interaction rate r and profile F. It is convenient to define $\psi_m(x) \equiv \mathbb{P}(\text{Poisson}(x) \geq m)$, and the functions

$$\Phi_k(x) \equiv \mathbb{E}[W\,\psi_{k-1}(Wx)]\,, \qquad A_k(x) \equiv \mathbb{E}[\psi_k(Wx)]\,. \tag{3}$$

When $W \equiv 1$ (the Erdős–Rényi benchmark), these reduce to $\Phi_k(x) = \psi_{k-1}(x)$ and $A_k(x) = \psi_k(x)$. Then define the critical cutoff

$$c_k(F) \equiv \inf_{x>0} \frac{x}{\Phi_k(x)}. \tag{4}$$

Under Assumption 1 and $k \geq 3$, $\Phi_k(x) = o(x)$ as $x \downarrow 0$ and $\Phi_k(x) \leq \mathbb{E}[W] = 1$ for all x, so $x/\Phi_k(x) \to \infty$ as $x \downarrow 0$ and as $x \to \infty$; hence the infimum in (4) is attained at some finite $x_k^\star(F) > 0$.[19] For expositional simplicity, we assume that the minimizer of $x/\Phi_k(x)$ is unique and satisfies a mild regularity condition.

Assumption 2. *$g(x) \equiv x/\Phi_k(x)$ has a unique minimizer and $g'(x) > 0$ for all $x > x_k^\star(F)$.*

This assumption ensures that the k-core emerges through a single discontinuous jump.[20] Our assumption always holds when the interaction profile is sufficiently homogeneous (such as the Erdős–Rényi benchmark; see Online Appendix OA.12.5 for the formal statement), and generally holds when heterogeneity is sufficiently "nice". In Online Appendix OA.12.2 we provide a single-crossing condition on F that is sufficient for Assumption 2 to hold. We also discuss the more general analogues to our main results. Unless explicitly stated otherwise, all results throughout the rest of the paper operate under Assumptions 1 and 2.

[18] For Erdős–Rényi graphs—which, in our model, corresponds to the homogeneous-interaction case—the canonical reference is Pittel et al. (1996); for configuration models with general degree distributions, see Janson and Luczak (2007); Janson and Łuczak (2008). The critical cutoff condition is usually referred to as the critical *threshold* in this mathematics literature. But to avoid confusion, we reserve the term "threshold" for the participation threshold k.

[19] For completeness, Lemma 4 in Online Appendix OA.12.1 shows $\Phi_k(x) = o(x)$ as $x \downarrow 0$.

[20] Without Assumption 2, g can have multiple local minima, in which case the giant k-core emerges through several disjoint jumps as connectivity increases. We discuss this case in Online Appendix OA.12.2.

Next, let $x_k(r)$ be the largest solution to

$$x = r\,\Phi_k(x) = r\,\mathbb{E}[W\,\psi_{k-1}(Wx)]\,. \tag{5}$$

Our first result shows that whether equilibrium participation is positive turns on whether $r \lesseqgtr c_k(F)$. The exact behavior of the finite graph at $r = c_k(F)$ depends on how it behaves in the "critical window" and is not important for our analysis. We therefore define limiting participation at the cutoff by its left limit:[21]

$$a_k\left(c_k(F); F\right) \equiv \lim_{r \uparrow c_k(F)} a_k(r; F) = 0.$$

With this in place, we are now ready to characterize equilibrium behavior.

Theorem 1 (Participation). *Fix $k \geq 3$ and an interaction profile F. Equilibrium participation:*

(i) is positive if and only if the interaction rate is sufficiently high ($r > c_k(F)$);

(ii) for every $r > c_k(F)$, is given by

$$a_k(r; F) = A_k(x_k(r)) = \mathbb{E}[\psi_k(W\,x_k(r))]\,, \tag{6}$$

where $x_k(r)$ is as in (5).[22]

Theorem 1 establishes a sharp tipping point in equilibrium behavior. Unless the interaction rate r exceeds the critical cutoff $c_k(F)$, the network is too sparse to support a giant k-core, and equilibrium participation is zero. Above this cutoff, a strictly positive fraction of agents participate, characterized by part (ii). As we show in Section 5, this transition is not gradual—participation *jumps* discontinuously from zero to a strictly positive level at the cutoff.

[21]This convention is without loss for the principal's limiting design problem. For every $\varepsilon > 0$, the interaction rate $r = c_k(F) - \varepsilon$ yields zero limiting participation, while the direct interaction payoff converges to its boundary value as $\varepsilon \downarrow 0$. Hence, in any environment where $r = c_k(F)$ generates a giant k-core with positive asymptotic probability, a principal with $\beta > 0$ would avoid this by choosing just below the cutoff at arbitrarily small direct-payoff cost. We therefore make no probabilistic claim about the finite graph at $r = c_k(F)$; all references below to choosing the cutoff should be read as choosing this "no participation boundary".

[22]Exactly at the cutoff $r = c_k(F)$, the fixed-point equation admits a tangency solution, but the k-core law of large numbers (Proposition 4.1 of Janson (2009)) does not apply there. Accordingly, we define $a_k(c_k(F); F) = 0$ by convention and study the right-limit $J_k(F) = \lim_{r \downarrow c_k(F)} a_k(r; F)$.

The intuition for this lies in the nature of social reinforcement required for participation. For any group of agents to form a stable, self-sustaining equilibrium (i.e., to be a k-core), each member must have their participation supported by at least k friends who are also part of that same group.

This stands in sharp contrast to the *continuous* emergence of the giant component—a related feature of large random networks that has been shown to play a key role in driving behavior in models where the network spreads information (see, for example, Dasaratha, 2023; Campbell et al., 2024). Unlike the giant k-core, the giant component is well studied in economics (see Jackson, 2008; Easley et al., 2010; Goyal, 2023, for textbook treatments).[23]

4.3 Comparative statics: how agents respond to the environment

We now consider how equilibrium participation varies with the interaction rate r and the participation threshold k. The level of participation is increasing in the interaction rate and decreasing in the participation threshold.

Theorem 2 (Comparative statics: monotonicity). *Fix an interaction profile F.*

(i) (Monotonicity in r.) *For every fixed $k \geq 3$, equilibrium participation, $a_k(r; F)$, is constant in r when $r < c_k$, and strictly increasing when $r > c_k$.*

(ii) (Monotonicity in k.) *For every fixed $r \geq 0$, equilibrium participation, $a_k(r; F)$, is constant in k when $r < c_k$, and strictly decreasing when $r > c_k$.*

Intuitively, as the interaction rate increases, the degree distribution shifts to the right—agents have more neighbors on average. This increases the likelihood that the number of an agent's neighbors who participate will exceed k, reinforcing participation. Conversely, a higher threshold k means fewer agents will have sufficient participating neighbors, causing them to abstain. This initial abstention can then induce other agents to do the same, ultimately reducing participation.[24] Formally,

[23] The giant component is closely related to the $k = 2$-core. Taking the giant component and removing all agents with degree 1 yields the 2-core. In the mixed-Poisson model that we use, the critical cutoff for the emergence of both the 2-core and the giant component is $r = 1/\mathbb{E}[W^2]$, which collapses to $r = 1$ in the Erdös-Rényi case.

[24] In contrast, when there is zero participation in equilibrium, changes in the network connectivity and/or the participation threshold have no impact. This follows immediately from Theorem 1(i).

these results are unlocked by our characterization in Theorem 1. To our knowledge, these comparative statics were not previously known.

A particular point of interest is the jump in participation due to the discontinuous emergence of the giant k-core. We now examine this jump in more detail. It is both of intrinsic interest—little is currently known about the jump beyond its existence—and will prove important for understanding the principal's optimal behavior.

5 Properties of the Jump

We now examine the location and size of the jump. Together these determine when participation first becomes possible and how much participation must arise when it does. We show that both are economically important, and characterize how they depend on the interaction profile F and on the participation threshold, k. To begin, we formally define the jump as the right-limit of participation at the critical cutoff:

$$J_k(F) \equiv \lim_{r \downarrow c_k(F)} a_k(r; F). \tag{7}$$

We first establish that the jump is indeed a jump. That is, above the critical cutoff, participation is bounded away from zero—no matter how close we get to the critical cutoff.

Proposition 1. *Fix any $k \geq 3$ and any interaction profile F.*

$$J_k(F) = A_k(x_k^*(F)) > 0.$$

As the interaction rate crosses the critical cutoff, participation jumps discontinuously from zero to a positive amount. Intuitively, this is because agents need *several* friends to take the action alongside them. When the interaction rate is low (below the critical cutoff), some set of agents may have at least k friends. And some subset of them may have at least k friends who in turn have at least k friends. But not enough of those friends will themselves have at least k friends who have k friends, and so on. There will be some "weak links" preventing mutually sustaining actions. So nobody participates. As the interaction rate rises just past the critical cutoff, those "weak links" will gain enough friends. This is the final piece of the puzzle for a significant group of agents—leading to this whole group now being able to sustain participation.

Hence participation jumps.

But is this group of agents large relative to the overall population? It turns out that this depends critically on the heterogeneity of the interaction profile, F.

5.1 The size of the jump

When interactions are homogeneous (i.e. when $W \equiv 1$), the jump involves at least one quarter of the whole population—it is *economically* significant. In fact, the jump is largest when the interaction profile is homogeneous, and is always reduced by heterogeneity.

Theorem 3 (Homogeneity maximizes the jump)**.** *Fix any $k \geq 3$. Let δ_1 denote the degenerate interaction profile with $W \equiv 1$.*
(i) (Homogeneous jump) *If $W \equiv 1$, then*

$$c_3 \approx 3.35, \quad \textit{and} \quad J_3(\delta_1) \approx 0.268.$$

(ii) (Homogeneous extremality) *For every interaction profile F,*

$$J_k(F) \ \leq \ J_k(\delta_1),$$

with equality if and only if $F = \delta_1$.

Theorem 3(i) formally characterizes the size and location of the jump when $k = 3$. We verify numerically that the size of the jump in the homogeneous case gets even larger as the participation threshold k rises. For example, when $k = 10$, $J_{10}(\delta_1) \approx 0.74$, and when $k = 100$ it is around 0.95 (see Figure 4).

Recent empirical work has identified a critical mass of approximately 25% as a "tipping point" required for a committed minority to overturn an established social norm (Centola et al., 2018). This work concerns the necessary conditions for *initiating* a large-scale behavioral cascade. Theorem 3 characterizes a complementary object—the conditions for the resulting behavior to be *self-sustaining*: under a moderate participation threshold ($k = 3$), any nonzero participation involves at least a quarter of the population in the homogeneous benchmark. Our result suggests a possible explanation for this empirical tipping point: individuals may be willing to adopt a new norm only when they perceive the movement has enough momentum to become self-sustaining.

Part (ii) is a global statement: it requires no restriction on the shape, support, or dispersion of F beyond those we already maintain. Homogeneity generates the largest possible jump, at every participation threshold. The proof of Theorem 3 provides an exact formula for the difference $J_k(\delta_1) - J_k(F)$. This formula shows that when the interaction profile is close to homogeneous, this difference is proportional to the variance of the interaction profile.[25] So interaction profiles with small amounts of heterogeneity have a jump close to the homogeneous case. We prove this formally in Online Appendix OA.10.

Greater heterogeneity shrinks the jump. We can go further and directly compare heterogeneous interaction profiles. To do this, we order them by how strongly interaction is concentrated among the most active agents. The quantile function Q associated with an interaction profile F is defined by $Q(u) \equiv \inf\{x \colon F(x) \geq u\}$. For two interaction profiles with quantile functions Q_1 and Q_2, say that F_2 is more concentrated than F_1 in the *star order* (Shaked and Shanthikumar, 2007),[26] written $F_1 \preceq_* F_2$, if there is a non-decreasing function $\rho : (0,1) \to [0,\infty)$ with

$$Q_2(u) \;=\; \rho(u)\, Q_1(u) \qquad \text{for almost every } u \in (0,1). \tag{8}$$

Moving up this order scales down the interaction weights of relatively inactive agents and scales up those of relatively active ones. For profiles with equal mean, the star order $F_1 \preceq_* F_2$ implies the convex order $F_1 \leq_{\mathrm{cx}} F_2$, but not conversely. Since the mean of our interaction profiles is always 1, moving up the star order corresponds to a special type of mean-preserving spread: one that shifts interaction weight toward more active agents.

The star order is particularly natural here because the homogeneous profile lies below every profile ($\delta_1 \preceq_* F$ for all F, by taking $\rho = Q_F$), and several natural parametric families—including gamma and lognormal—are ordered this way as functions

[25] For example, when $k = 3$, $J_k(\delta_1) - J_k(F) \approx 0.78\,\mathrm{Var}(W)$. See Remark 5 in Online Appendix OA.10 for details. Because $\mathrm{Var}(W)$ is identified by the mean and variance of realized degrees (see Section 7), this provides a way to approximate the jump size from aggregate data on real networks.

[26] The multiplicative form (8) is a natural adjustment of the usual star order to handle distributions with mass at zero correctly: an agent with zero weight under F_1 has zero weight under F_2. A definition based on monotonicity of the quantile ratio wherever it is defined would misclassify profiles that only add mass at zero. Our definition is equivalent to the usual star order when distributions are atomless with strictly positive support.

of their variance.

Theorem 4 (Jump comparison)**.** *Fix $k \geq 3$ and let $F_1 \preceq_* F_2$. Then*

$$J_k(F_2) \;\leq\; J_k(F_1),$$

with equality if and only if $F_1 = F_2$.

The jump is therefore *decreasing in concentration*: decreasing the interaction of the "inactive many" while increasing the interaction of the "active few" pushes down the size of the jump. Theorem 3(ii) is the special case $F_1 = \delta_1$: the homogeneous profile lies at the bottom of every chain, so its jump is always largest. The intuition is as follows: at the point where the k-core first appears there are many agents who expect close to k participating neighbors, and concentration moves interaction weight away from these marginal agents toward agents already safely inside the core—shrinking the share of the population the core contains.

Moreover, there need be no end to this "concentration" logic. It is possible to continually shrink the fraction of agents in the "interactive few", increasing their interaction weights W_i ever higher, while correspondingly shrinking the fraction of agents in the "inactive many". As a result, it is possible to make the jump arbitrarily small by introducing sufficient heterogeneity. We provide a parametric example in Section 5.3 that illustrates this limiting argument explicitly.

The impact of the participation threshold. In contrast, the size of the jump need not always be increasing in the participation threshold k.[27] However, numerical methods show that for natural parametric families, the size of the jump is increasing and concave. For example, in the homogeneous benchmark, $J_3(\delta_1) \approx 0.27$, $J_{10}(\delta_1) \approx 0.74$, and $J_{100}(\delta_1) \approx 0.95$. Numerically, $J_k(\delta_1)$ asymptotes toward 1 at high k. The heterogeneous profiles we plot in Figure 4 exhibit the same monotonicity but generally at lower levels, with the apparent asymptote strictly below 1.

5.2 The location of the jump

We now consider how heterogeneity in interactions affects the location of the cutoff—the level of connectivity needed for any collective action to be sustained. The cutoff is

[27] See Online Appendix OA.7 for an explicit example.

not ranked by the star order that ranks the size of the jump. So greater heterogeneity (in the sense defined by the star order) can move the cutoff in either direction. But we can provide comparative statics on the location of the cutoff using a different strengthening of the convex order.

A function $\phi\colon [0,\infty) \to [0,\infty)$ is said to be *starshaped* if $\phi(0) = 0$ and $\phi(w)/w$ is nondecreasing. Following Shaked and Shanthikumar (2007, §4.A.6), say that F_1 is below F_2 in the *starshaped order*, written $F_1 \leq_{\mathrm{ss}} F_2$, if $\mathbb{E}[\phi(W_1)] \leq \mathbb{E}[\phi(W_2)]$ for every starshaped function ϕ.[28] Under this partial ordering, we can unambiguously rank the location of the cutoff for different interaction profiles.

Proposition 2 (Cutoff comparison)**.** *Fix $k \geq 3$. If $F_1 \leq_{\mathrm{ss}} F_2$, then $c_k(F_2) \leq c_k(F_1)$.*

The proof (in Section A.6) is only a few lines long. For every x, the function $w \mapsto w\,\psi_{k-1}(wx)$ inside Φ_k is itself starshaped, so moving up the starshaped order raises Φ_k pointwise and pushes the cutoff $c_k = \inf_x x/\Phi_k(x)$ down. The same argument does not apply to the jump, as it is not determined by a starshaped function. In network terms, the starshaped order is equivalent to first-order stochastic dominance of the degree distribution over vertices obtained by choosing a node at random and following it to one of its neighbors. For mean-one distributions, moving up the starshaped order requires a strictly larger share of non-interacting agents—two distinct profiles with strictly positive support are never comparable (Online Appendix OA.9)—so the order captures concentration of the core–periphery kind.

Although both the star order and the starshaped order are refinements of the convex order, they capture distinct changes in the distribution over interactions. The star order ranks the jump (Theorem 4) but not the cutoff; the starshaped order ranks the cutoff (Proposition 2) but not the jump. The convex order ranks neither.[29]

The impact of the participation threshold. Unlike the size of the jump, the location of the jump—the cutoff—is unambiguously increasing in the participation

[28] Despite the similarity between "star" and "starshaped", these two orders are in fact unrelated: the star order compares quantile functions, the starshaped order compares expectations of test functions, and neither implies the other (Online Appendix OA.9). For mean-one profiles the starshaped order is equivalent to first-order stochastic dominance of the *size-biased* weight (Shaked and Shanthikumar, 2007, Theorem 4.A.54)—the weight of a randomly chosen neighbor. This is the continuous analogue of the "dominance in expectation" order on degree distributions in Sadler (2020).

[29] We provide examples in Online Appendix OA.9 where, for example, star-ordered profiles can move the cutoff in opposite directions, and likewise for the jump.

threshold k. Intuitively, when each agent needs more participating friends, the network must be more connected to sustain participation at all. This increases the cutoff.

Proposition 3. *For any interaction profile F, the critical cutoff $c_k(F)$ is strictly increasing in k.*

Numerically, the cutoff is approximately linear in k in the parametric interaction profiles that we consider (see Figure 4 below). For the homogeneous interaction profile, asymptotic linearity in k can be proved analytically: $c_k(\delta_1) \sim k$ as $k \to \infty$ (Pittel et al., 1996).

5.3 Parametric Examples

Our analytic results on how the size and location of the jump vary with the interaction profile are powerful, but provide only partial orderings. To obtain sharper comparative statics and examine the two objects jointly, we now consider several parametric environments. We first consider a stark core-periphery structure, which we call a "dilution". There, the jump size and location respond to heterogeneity in a particularly simple way. We then show that the qualitative insights from that setting extend to the gamma and lognormal distributions—two common families of parametric distributions.

Definition 1. *For any interaction profile F with weights W, we define a* diluted *interaction profile, $F^{(\nu)}$ for any $\nu \in (0,1]$ with weights*

$$W^{(\nu)} \equiv \frac{B}{\nu} W,$$

where $W \sim F$ and $B \sim \mathrm{Bernoulli}(\nu)$ are independent.

In a dilution of any interaction profile F, a fraction $1-\nu$ of agents do not interact at all, and the remaining fraction ν have their interaction rate scaled up by a factor $1/\nu$. This creates a stark *core–periphery* structure: a small, highly active core and an inactive periphery. In this special case, both summary statistics scale linearly in ν.

Remark 2. *If F satisfies Assumption 2, then so does $F^{(\nu)}$, and for all $k \geq 3$*

$$c_k(F^{(\nu)}) = \nu\, c_k(F) \qquad \text{and} \qquad J_k(F^{(\nu)}) = \nu\, J_k(F).$$

Bernoulli dilution is especially well behaved under the distributional orders introduced above. From any starting interaction profile, greater dilution (lower ν) moves it up the starshaped order. When the starting profile is homogeneous, dilution also moves it up the star order. Thus, our general comparative statics already show that greater dilution results in both a lower cutoff and, in the homogeneous case, a smaller jump. Remark 2 strengthens these results by showing that the jump is ordered under dilution from *any* starting profile, and that both quantities scale linearly with the "active" fraction ν.

Dilution shows that heterogeneity can shrink the size of the jump not by changing the structure of interactions within the core, but rather by restricting the set of agents who can plausibly form a mutually reinforcing group. The network may be extremely dense *within* a small active set (mean degree r/ν), while the overall participation fraction scales with the size ν of that set. Moreover, in this environment, more heterogeneity induces a lower cutoff: $c_k(F^{(\nu)}) = \nu\, c_k(F)$. As the active fraction shrinks, participation becomes possible at lower interaction rates—the network needs less overall connectivity to trigger collective action among the remaining active agents.[30]

Gamma and lognormal distributions. In the gamma and lognormal distributions, heterogeneity is naturally captured by variance. Figure 3 plots the critical cutoff c_k and the size of the jump J_k as a function of the dispersion $\sigma(F) = \sqrt{\mathrm{Var}(W)}$ for both families at $k = 3$. For these families, greater heterogeneity shrinks the size of the jump. When $k = 3$, greater dispersion also lowers the cutoff.

[30] In Online Appendix OA.5 we show that Remark 2 is robust to relaxing the assumption that peripheral agents have no links.

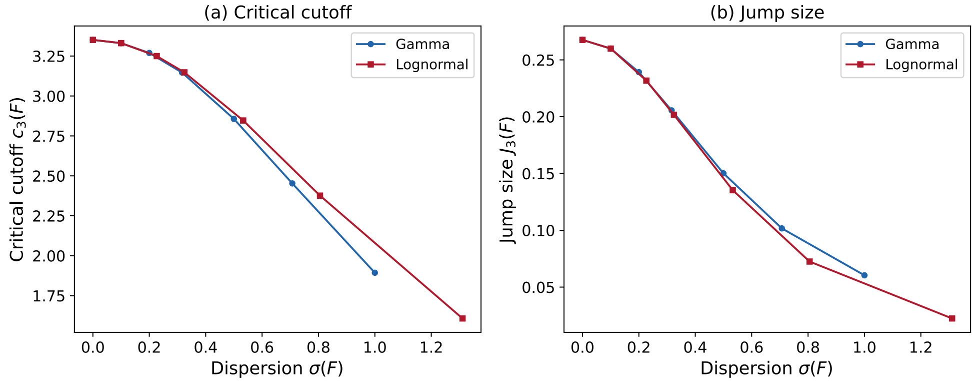

Figure 3: Critical cutoff $c_3(F)$ (left) and jump $J_3(F)$ (right) against dispersion $\sigma(F) = \sqrt{\mathrm{Var}(W)}$, for the gamma and lognormal families. Both families show the same pattern at $k = 3$: increasing heterogeneity lowers the cutoff and shrinks the jump.

In fact, for the size of the jump, the pattern shown in Figure 3 follows directly from Theorem 4: within both the gamma and lognormal families increasing the variance moves the interaction profile up in the star order. So J_k is strictly decreasing in variance along these families (whenever they satisfy Assumption 2), with $J_k \uparrow J_k(\delta_1)$ as the variance goes to 0.[31]

The impact of the participation threshold. Figure 4 plots both c_k and J_k in the homogeneous benchmark, the gamma family, and the core–periphery environment up to $k = 150$. We take $k = 150$ as the upper limit because it is a common benchmark for the size of an individual's typical full social network (Dunbar, 1998). For behaviors driven by reinforcement from close peers, the relevant number is far smaller. According to representative U.S. survey data, the average number of people with whom Americans discuss "important matters" is around two (McPherson et al., 2006). Relatedly, experimental work on complex contagion finds large effects of the first two or three neighbors adopting, and diminishing effects thereafter (Centola, 2010).[32]

[31]Every gamma-distributed interaction profile satisfies Assumption 2. Numerical methods indicate that Assumption 2 holds for the lognormal family also, though we cannot prove this analytically.

[32]The Bernoulli dilution case is omitted from the figure: Remark 2 provides exact scaling identities $c_k(F^{(\nu)}) = \nu\, c_k(F)$ and $J_k(F^{(\nu)}) = \nu\, J_k(F)$, so the shape in k under dilution inherits directly from the underlying profile F.

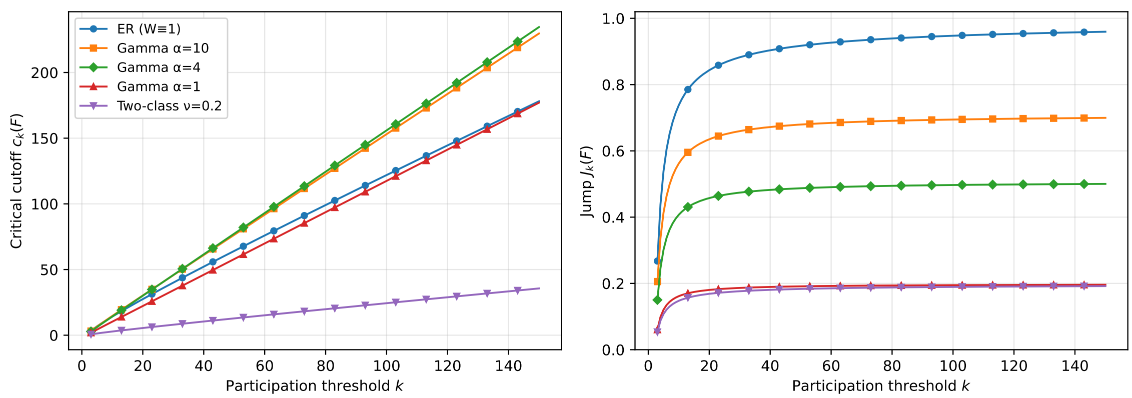

Figure 4: Critical cutoff $c_k(F)$ (left panel) and jump size $J_k(F)$ (right panel) as functions of the participation threshold k, for the homogeneous benchmark, the gamma family, and the core–periphery environment. The cutoff rises monotonically and approximately linearly in k; the jump rises monotonically with concave shape, with limiting value 1 in the homogeneous benchmark and lower values for the heterogeneous families.

The two plots carry distinct economic implications for the principal's design problem. A higher k pushes the cutoff up, giving the principal more room to raise r while still enjoying zero participation. It also amplifies the jump, making movement just above the cutoff more costly. Together, these make the zero-participation regime more attractive at higher k. Working in the opposite direction, Theorem 2(ii) says that when $r > c_k$, increasing k reduces participation, which all else equal pushes the principal toward a higher interaction rate. Section 6 examines the resulting trade-off and shows that the principal's optimal interaction rate can move in either direction in response to a change in k. We now turn to the principal's problem.

6 How the Principal Designs the Network

The principal's problem is to choose the interaction rate r to maximize her payoff (1), taking as given how agents will behave in the second stage—which is in turn a function of the participation threshold, k, and the interaction profile, F. Substituting the equilibrium participation $\bar{a} = a_k(r; F)$ from Section 4, she maximizes

$$\pi(r) = \alpha r - \tfrac{1}{2}\, r^2 - \beta\, a_k(r; F). \tag{9}$$

The headline result is that if the interaction profile is homogeneous, the principal will never choose an interaction rate that is close to but above the critical cutoff, $c_k(F)$. This is precisely due to the large jump in the level of participation (i.e. the discontinuous emergence of the giant k-core) at the critical cutoff—the qualitative results do not depend on the particular functional form of the principal's payoffs. Setting the interaction rate exactly at the cutoff delivers no participation at all. But going even a very small amount above the cutoff suddenly yields significant participation. This creates significant costs for the principal, but only small incremental benefits. So she is better off by either "returning" to the critical cutoff, or "pushing through" to a much higher interaction rate—generating large enough direct benefits from interactions to offset the participation costs.

We call this push to extreme choices the "missing middle". But note that the missing middle is only economically meaningful when the jump is large. In particular, highly heterogeneous interaction profiles can generate small jumps at the critical cutoff, and hence small losses from choosing an interaction rate just above that cutoff.

We also examine how changes in parameters alter the principal's optimal choice. A higher cost to participation, β, leads the principal to choose a lower interaction rate, and can induce a switch to complete suppression of participation. In contrast, an increase in the participation threshold, k, can go either way. While it reduces participation for any fixed interaction rate (Theorem 2(ii)), the principal may respond by increasing the interaction rate by so much that it overwhelms this effect.

6.1 Choosing the optimal interaction rate

Preliminary observations. To start, let $r^{\text{naive}} = \alpha$. This is what the principal would choose absent any concerns about participation (i.e. $\beta = 0$). And because the level of participation, $a_k(r; F)$, is weakly increasing in the interaction rate, the principal never chooses an interaction rate above this level (i.e. $r > r^{\text{naive}}$). Next, note that since π is upper semicontinuous and non-negative on a compact interval (due to the convex costs), it attains its maximum on that interval.Moreover, the optimum is generically unique. This is because the value function $V(\beta) \equiv \sup_{r \geq 0} \pi(r; F)$ is convex in β (as the pointwise supremum of affine functions), hence differentiable except at a countable set. At any β where V is differentiable, the global maximizer is unique (we prove this in Online Appendix OA.12.8). So if there are two or more optimal choices of the interaction rate then a small perturbation of β restores uniqueness.

Finally, notice that the problem is trivial if $\alpha \leq c_k(F)$. In this case, the principal's unconstrained optimum, $r^{\text{naive}} = \alpha$, lies *below* the critical cutoff and triggers no participation. So throughout this section we focus on the case where $\alpha > c_k(F)$. This is where the principal faces a genuine tension between the benefits of interaction and the costs of the participation it induces.

The missing middle. The discontinuity in equilibrium participation creates a distinctive feature of the principal's problem. Because participation jumps discretely from zero to at least $J_k(F)$ when r crosses the critical cutoff $c_k(F)$, the principal will never choose an interaction rate "just above" the cutoff. If she moves even slightly beyond $c_k(F)$, she incurs a participation cost of at least $\beta J_k(F)$—a discrete penalty—while the direct benefit $\alpha r - \frac{1}{2}r^2$ improves only marginally relative to its value at $c_k(F)$. For each $\beta \geq 0$, define

$$\pi_\beta(r;F) \equiv \alpha r - \frac{1}{2}r^2 - \beta\, a_k(r;F).$$

The cutoff participation cost is

$$\bar{\beta}(\alpha;F) = \sup\left\{\beta > 0 : \sup_{r > c_k(F)} \pi_\beta(r;F) \geq \pi_\beta\big(c_k(F);F\big)\right\}. \tag{10}$$

With this notation in place, we can now formalize the "missing middle" intuition.

Proposition 4. *Suppose $\alpha > c_k(F)$, and let $\bar{\beta}(\alpha;F)$ be defined in* (10)*.*

(i) (High participation costs.) *if $\beta > \bar{\beta}$, the optimal interaction rate is exactly at the critical cutoff (and the level of participation is zero): $r^\star = c_k(F)$.*

(ii) (Low participation costs.) *if $\beta < \bar{\beta}$, the optimal interaction rate is at least a distance $d_k(F) > 0$ above the critical cutoff: $r^\star \geq c_k(F) + d_k(F)$, where*

$$d_k(F) = \frac{\beta\, J_k(F)}{\alpha - c_k(F)} > 0. \tag{11}$$

The result splits optimal behavior into two regimes, with a sharp boundary at $\bar{\beta}$. When participation is sufficiently costly for the principal, she will choose the highest interaction rate that yields no participation. And when participation is not too costly, she will choose an interaction rate significantly above the critical cutoff, accepting a high level of participation in the process.[33]

The term $d_k(F)$ is a lower bound on how far above the critical cutoff the principal must move to recover her loss from the jump. It of course depends on the parameters α and β. But its dependence on the rest of the environment is entirely through the critical cutoff $c_k(F)$ and the jump $J_k(F)$—the two key summary statistics from Section 4. Section 5's comparison results discipline both of these statistics, each through its own order. The jump is ranked across environments: no interaction profile's jump exceeds the homogeneous benchmark (Theorem 3), and more concentrated profiles have strictly smaller jumps (Theorem 4). Through the numerator, the gap is therefore bounded by the worst-case jump: $d_k(F) \leq \beta J_k(\delta_1)/\big(\alpha - c_k(F)\big)$. The cutoff is not ranked by the star order, but it is ranked by the starshaped order (Proposition 2), and it enters the gap through the denominator: across environments, a lower cutoff widens $\alpha - c_k(F)$ and narrows the gap further. The two channels operate together in the core–periphery family: the jump and the cutoff scale down in step (Remark 2), and the missing middle unambiguously narrows—$d_k(F^{(\nu)}) \leq d_k(F)$ for every $\nu \in (0, 1]$.[34]

Heterogeneity in interactions therefore determines the size of the missing middle through the jump. In homogeneous environments, $J_k(F)$ is large, so the missing middle is large. In contrast, more heterogeneous environments have a small jump, which weakens the discrete tradeoff near the cutoff and makes an interaction rate closer to the critical cutoff less costly.[35]

The missing middle also creates a sensitivity in the principal's optimal choice of r. Although any equilibrium with positive participation has at least $J_k(F)$ participants, small changes in the incentives (α or β) may induce the principal to "flip" from an

[33] The cutoff $\bar{\beta}(\alpha; F)$ is a knife-edge in the middle, where there is both an optimal choice at the critical cutoff and one significantly above it. The principal is of course indifferent between these choices.

[34] This follows immediately from the scaling: $d_k(F^{(\nu)}) = \beta\nu J_k(F)/(\alpha - \nu c_k(F))$, and $\nu\,(\alpha - c_k(F)) \leq \alpha - \nu\, c_k(F)$ since $\nu \leq 1$.

[35] Of course, the global optimum may lie well above $c_k(F_\nu)$. This will depend on the parameters (α, β). Proposition 4 characterizes a region above the cutoff $c_k(F)$ that can *never* be optimal. Additionally, note that our analysis has focused on the case where there is only one jump in the size of the giant k-core (this is the case where the function g has a unique minimizer), and hence in the level of participation. If participation has additional jumps at higher r, an analogous missing middle can arise there; we leave that extension for future work.

equilibrium with substantial participation to one with zero (or vice versa). To explore this sensitivity, we now describe how the principal's equilibrium choice of r depends on the participation cost β.

6.2 Comparative statics and tipping points

The next result gives both a robust (set-valued) comparative static and a stronger derivative statement at regular interior optima. Let $R(\beta) \equiv \arg\max_{r \geq 0} \pi_\beta(r; F)$ denote the set of optimal interaction rates, and let $\bar{a}^\star \equiv a_k(r^\star; F)$ denote the equilibrium participation at the optimal policy.

Proposition 5 (Comparative statics and tipping points)**.** *Fix $\alpha > c_k(F)$ and let $\bar{\beta} = \bar{\beta}(\alpha; F)$.*

(i) (Monotonicity.) *If $0 \leq \beta_1 < \beta_2$ and $r_i \in R(\beta_i)$, then $r_2 \leq r_1$.*

(ii) (Strict monotonicity.) *Moreover, if $0 \leq \beta_1 < \beta_2 < \bar{\beta}$ and each $R(\beta_i)$ is a singleton $\{r_i\}$, then $r_2 < r_1$ and $a_k(r_2; F) < a_k(r_1; F)$.*

(iii) (Local derivative.) *At any $\beta < \bar{\beta}$ where the maximizer $r^\star$ is unique, lies strictly above $c_k(F)$, and satisfies $\frac{\partial^2}{\partial r^2}\pi_\beta(r^\star; F) < 0$,*

$$\frac{\mathrm{d}r^\star}{\mathrm{d}\beta} < 0 \quad \textit{and} \quad \frac{\mathrm{d}\bar{a}^\star}{\mathrm{d}\beta} < 0.$$

Proposition 5 shows the difference between the two regimes. When the principal tolerates the behavior in equilibrium ($\beta < \bar{\beta}$), adjustments are smooth: as β increases, the principal reduces the interaction rate. This, in turn, leads to a reduction in participation. In contrast, when the optimal strategy is to completely suppress participation ($\beta > \bar{\beta}$), the principal's choice of $r^\star$ is "sticky". Small changes in her incentives are insufficient to move the interaction rate away from the critical cutoff; she continues to choose the highest possible interaction rate that guarantees zero participation.

The key takeaway from this result is that when β is close to $\bar{\beta}$, a small change in the cost of participation can move her from choosing an equilibrium with positive participation to one with zero participation (or the reverse). This highlights a tipping-point phenomenon in the principal's decision-making: her strategy, and more importantly the resulting participation, can shift dramatically in response to relatively mild changes in her incentives.

Raising the threshold. A higher participation threshold k exerts two opposing forces on the principal's choice. For any fixed $r > c_k$, it decreases the participation $a_k(r; F)$ (Theorem 2(ii)), pushing the principal toward higher r; but it also raises c_k and—for the interaction environments we plot in Figure 4—the size of the jump $J_k(F)$, making it more attractive to fully suppress participation. In the cases we have examined numerically, equilibrium participation $\bar{a}^\star$ is always weakly decreasing in k, while the optimal interaction rate $r^\star$ can move in either direction: Online Appendix OA.6 provides examples that go both ways. We leave a sharper characterization for future work.

7 Concluding Remarks

We finish by discussing the settings beyond protest in which the tension we study appears. We then comment briefly on some of our modeling choices and how they relate to directions for future research.

Beyond protest. Protest against a government is our leading application, but the same trade-off appears wherever a principal values interaction that can also sustain collective behavior she dislikes. Firms shape how much their workers interact—through office design, communication technology, and remote-work policy (Bernstein and Turban, 2018; Yang et al., 2022). That interaction raises productivity (Mas and Moretti, 2009), but it can also sustain misconduct (Dimmock et al., 2018) and unionization (Dale-Olsen et al., 2025). Platforms face the trade-off most directly: their product is connection, yet connection can amplify misinformation, and WhatsApp's cap on message forwarding is a prominent example of a platform deliberately reducing aggregate connectivity (Melo et al., 2020). In each case, whether the principal is pushed to an extreme response depends not only on her preferences, but on how interaction is distributed across the population she governs.

Strategic complementarities. The model's core mechanism is a participation threshold driven by strategic complementarities. We interpret this broadly as a process of social reinforcement, where an agent's incentive to act is fundamentally linked to the actions of his direct network connections. The parameter k represents the critical mass of local support required to make an action worth taking. In our setting, this threshold is an *absolute number* of neighbors, rather than a fraction of all neighbors. This is technically useful, as it allows us to build on existing work in random graph theory. We view an absolute threshold as appropriate in settings where agents need a certain "critical mass" of local support for an action in order to be willing to participate. For example, protest participation garners "safety in numbers" only when enough of an individuals friends participate, regardless of total network size. Likewise, adopting a new norm may be beneficial only when a minimum number of peers adopt it too. Settings where agents instead care about the *share* of neighbors taking an action—such as language choice, where speakers switch when most of their contacts have adopted the new language—would benefit from a different modeling approach.

The random network. Formally, our random network is a mixed-Poisson model.[36] In the configuration model, agents form links uniformly at random. This is a deliberate assumption to isolate the interplay between complex contagion and network heterogeneity in expected degrees (i.e. agents' propensities to form links). However, our model abstracts from several important features of real-world networks. These include homophily—the tendency for agents to form links with similar others—and multiplexity, the idea that different types of links can play different roles. Additionally, agents in our framework do not make endogenous decisions about forming or severing links; the network is determined by the principal's choice of interaction rate and the realization of the random graph. In reality, agents may have some control over both how many connections to form and with whom to form them.

Relaxing these assumptions provides a clear agenda for future work. Incorporating homophily into the configuration model—for instance through a stochastic block model with heterogeneous degrees—would allow the analysis to capture group-level variation in both connectivity and behavior. Endogenous link formation, where agents can respond to the principal's choice by rewiring their connections, represents a more significant challenge but is important for applications where agents strategically

[36] This nests the canonical Erdős–Rényi network as a special case when $W \equiv 1$. When $W \equiv 1$, agents' degrees are i.i.d. Poisson(r), which is asymptotically equivalent to $G(n, r/n)$.

manage their exposure to contagion.

The principal's levers. In our setting, the principal's policy levers are summarized by a single parameter, the interaction rate r. In practice, this may reflect a bundle of decisions—such as the frequency or size of public gatherings, the access to communication infrastructure, and the scope of curfews or other mobility restrictions—that collectively shape how much agents interact. But we are considering a setting where the principal lacks more fine-grained tools required to target specific groups of agents, or induce agents to interact with specific people. Hence she treats the interaction profile F as fixed. A natural extension is a joint dynamic problem in which policy choices can also shape F; we leave that extension for future work.

Another natural tool for the principal is targeted interventions—such as degree-dependent subsidies or selective seeding of agents to shift their behavior. In our framework, random seeding can be analyzed cleanly via Poisson thinning, but the interaction between seeding and the endogenous adjustment in connectivity by the principal is more interesting: when the principal can respond to seeding by adjusting r, the net effect on participation may depend on the interaction profile. We view this as another promising direction for future work.

An advantage for empirical work. Despite the abstractions, our framework's tractability offers a practical advantage: the key tradeoffs are driven by the interaction rate r and the interaction profile F, rather than by the full network adjacency matrix. This makes the model's predictions testable with aggregate data on connectivity patterns, and provides a useful benchmark for richer models that incorporate more detailed network features.

Moreover, the heterogeneity in our interaction profile F is directly tied to degree moments. The variance in realized degrees depends only on the interaction rate r and the variance in the weights W_i. Formally: $\mathrm{Var}(D_i) = \mathbb{E}[\mathrm{Var}(D_i \mid W_i)] + \mathrm{Var}(\mathbb{E}[D_i \mid W_i]) = r + r^2\mathrm{Var}(W_i)$. So the variance in our (unobservable) interaction profile F is identified by the mean and variance of the realized degree distribution.

Wrapping up. This paper sets out a tractable framework for analyzing complex contagions on heterogeneous networks. By using a random networks approach we are able to provide clean comparative statics for how aspects of network structure affect equilibrium behavior. Notably, we show that a sudden jump in behavior around a critical level of network connectivity arises directly from the threshold-based nature of complex contagion, and that heterogeneity in the network structure influences the size and location of this jump.

These results then unlock an analysis of a principal's optimal design problem, where she can choose network connectivity but not the finer structure of the network. When the network structure is homogeneous, the principal is forced into extreme all-or-nothing policies. She either holds connectivity at the critical cutoff to prevent any participation or sets connectivity well above it, allowing significant participation. There is no middle ground. This is driven by the jump. If the principal accepts any participation, she must accept significant participation, so she is "in for a penny, in for a pound."

But the homogeneous benchmark is also the worst case: no interaction profile produces a larger jump, and any heterogeneity strictly dampens it. In environments where interaction is concentrated in a small active core, the jump—and with it, the force driving extreme policy—becomes arbitrarily small, and more moderate policies can arise.

Together, these insights provide a new link between network heterogeneity and the optimal policy of a principal with coarse control over the network. Our results explain why governments in some environments resort to extreme measures (such as internet shutdowns) while in others, the pressure toward extremes is weaker and moderate policies can emerge.

A Appendix A: Proofs

A.1 Proof of Theorem 1 (Participation)

Proof. The case $r = 0$ is immediate, so fix $r > 0$. We apply Proposition 4.1 of Janson (2009). Following his notation, write Core_k^* for the k-core of the multigraph configuration model and $v(\mathrm{Core}_k^*)$ for its number of vertices. The proposition applies to a sequence of deterministic degree sequences satisfying his Condition 2.1; as usual, the total degree must be even.

Let D_i denote the degree of agent i. Conditional on W_i, $D_i \mid W_i \sim \mathrm{Poisson}(rW_i)$ independently across i. Unconditionally, the D_i are i.i.d. with mixed-Poisson distribution

$$p_j(r;F) \equiv \mathbb{P}(D_i = j) = \mathbb{E}_W\left[e^{-rW}\frac{(rW)^j}{j!}\right], \qquad j \geq 0,$$

and mean $\mathbb{E}[D_i] = r$. Let $n_j^{(n)} \equiv \#\{i \leq n : D_i = j\}$. By the strong law of large numbers,

$$\frac{n_j^{(n)}}{n} \to p_j(r;F) \qquad \text{for every } j \geq 0, \qquad \frac{1}{n}\sum_{i=1}^n D_i \to r$$

almost surely. These are exactly conditions (2.1) and (2.2) of Janson (2009): the empirical degree distribution converges to $(p_j(r;F))_{j\geq 0}$, and its mean converges to $\sum_j j\, p_j(r;F) = r$. Conditional on a realized degree sequence satisfying these limits, Proposition 4.1 therefore applies to the corresponding multigraph configuration model. Its convergence conclusions carry over to the simple-graph-conditioned model used in Section 3.[37] Write Core_k for the k-core in this conditioned simple graph.

Let D have the limiting degree distribution $(p_j(r;F))_{j\geq 0}$, and let D_p denote its p-thinning; that is, $D_p \sim \mathrm{Bin}(D,p)$. Proposition 4.1 characterizes the k-core using

$$h_1(p) \equiv \mathbb{P}(D_p \geq k) \qquad \text{and} \qquad h(p) \equiv \mathbb{E}[D_p\,\mathbf{1}_{\{D_p \geq k\}}].$$

Let $\widehat{p} \equiv \max\{p \in [0,1] : h(p) = rp^2\}$. Proposition 4.1 gives $v(\mathrm{Core}_k^*)/n \xrightarrow{p} 0$ if $\widehat{p} = 0$, and $v(\mathrm{Core}_k^*)/n \xrightarrow{p} h_1(\widehat{p}) > 0$ if $\widehat{p} > 0$ and is not a local maximum of $h(p) - rp^2$. Thus h determines whether a giant k-core exists, while $h_1(\widehat{p})$ gives its limiting size.

[37] Equation (1.1) of Janson (2009) states that the configuration model is simple with probability bounded away from zero if and only if $\sum_i D_i^2 = O(\sum_i D_i)$. Under Assumption 1, $\mathbb{E}[D_i^2] = r + r^2\mathbb{E}_W[W^2] < \infty$. The strong law therefore gives $\sum_i D_i^2 = O(n)$ and $\sum_i D_i \sim rn$ almost surely, verifying this criterion.

We now translate these quantities into our notation. Conditional on W, $D_p \mid W \sim \mathrm{Poisson}(rWp)$. Therefore

$$h_1(p) = \mathbb{E}_W[\psi_k(rWp)] = A_k(rp),$$

where the last equality follows from (3). It is straightforward to show that, if $N \sim \mathrm{Poisson}(\lambda)$, then $\mathbb{E}[N\,\mathbf{1}_{\{N\geq k\}}] = \lambda\,\psi_{k-1}(\lambda)$. Applying this identity to $D_p \mid W$ gives $\mathbb{E}[D_p\,\mathbf{1}_{\{D_p\geq k\}} \mid W] = rWp\,\psi_{k-1}(rWp)$. Taking expectations,

$$\begin{aligned} h(p) &= \mathbb{E}_W[rWp\,\psi_{k-1}(rWp)] \\ &= rp\,\Phi_k(rp). \end{aligned}$$

Since $p = 0$ always solves $h(p) = rp^2$, only positive roots can generate a giant k-core. For $p > 0$, substituting the preceding expression for h and setting $x = rp$ reduces this root equation to

$$x = r\Phi_k(x).$$

Thus its positive roots correspond exactly to the positive solutions of our fixed-point equation, and $h_1(p) = A_k(x)$ at the corresponding value $p = x/r$.

Recall that $c_k(F) = \inf_{x>0} g(x)$, where $g(x) = x/\Phi_k(x)$. If $r < c_k(F)$, then $g(x) > r$ for every $x > 0$, so the root equation has no positive solution. Proposition 4.1(i) of Janson (2009) then gives $v(\mathrm{Core}_k)/n \xrightarrow{p} 0$.

Now suppose $r > c_k(F)$. By Assumption 2, the function $g(x) = x/\Phi_k(x)$ has a unique minimizer $x_k^\star(F)$ and is strictly increasing on $[x_k^\star(F), \infty)$. Let $x_k(r)$ denote the largest positive solution to $x = r\Phi_k(x)$. Then $x_k(r) > x_k^\star(F)$. Choose $\delta \in (0, x_k(r) - x_k^\star(F))$. For every $x \in (x_k(r) - \delta, x_k(r))$, $g' > 0$ implies $g(x) < g(x_k(r)) = r$; that is, $x < r\Phi_k(x)$. Writing $p = x/r$, we obtain

$$h(p) - rp^2 = rp\big(\Phi_k(rp) - p\big) > 0 \qquad \text{for all } p \in \big((x_k(r) - \delta)/r,\, x_k(r)/r\big).$$

Since equality holds at $\widehat{p} = x_k(r)/r$, this is exactly the condition in Proposition 4.1(ii) of Janson (2009) that the largest positive root is not a local maximum of $h(p) - rp^2$. Therefore

$$\frac{v(\mathrm{Core}_k)}{n} \xrightarrow{p} A_k(x_k(r)) > 0.$$

By Remark 1, in every realized network the fraction of participating agents equals

$v(\mathrm{Core}_k)/n$, which gives (6) for $r > c_k(F)$.

At the exact critical cutoff $r = c_k(F)$, Proposition 4.1 of Janson (2009) does not apply. We make no claim about the finite-n k-core at this point. We adopt the convention $a_k(c_k(F); F) = 0$ which is without loss for our purposes—see footnote 21 in the main text. □

A.2 Proof of Theorem 2 (Comparative statics)

The following elementary observations are used repeatedly in the proof below. We therefore collect them in a Remark.

Remark 3 (Poisson-tail monotonicity)**.** *Since $W \geq 0$ and $\mathbb{E}[W] = 1$, we have $\mathbb{P}(W > 0) > 0$. For every integer $m \geq 1$, $\psi_m(y)$ is strictly increasing in $y > 0$, while $\psi_{m+1}(y) < \psi_m(y)$ for every $y > 0$. It follows that A_k and Φ_k vanish at zero and are strictly increasing on $(0, \infty)$. Moreover, for every $x > 0$,*

$$A_{k+1}(x) < A_k(x) \qquad \textit{and} \qquad \Phi_{k+1}(x) < \Phi_k(x).$$

Proof. For every $k \geq 3$ and $r \geq 0$, write $T_{k,r}(x) \equiv r\Phi_k(x)$. The map $T_{k,r}$ is continuous and increasing in x. Moreover, $T_{k,r}(x) = r\Phi_k(x) \leq r$ for every $x \geq 0$ since $\Phi_k(x) \leq \mathbb{E}[W] = 1$, so $T_{k,r}$ maps $[0, \infty)$ into $[0, r]$. By definition, $x_k(r)$ is its largest fixed point.

(i) Monotonicity in r. Fix the participation threshold k. For $r \leq c_k(F)$, Theorem 1 gives $a_k(r; F) = 0$, so monotonicity is immediate whenever $r \leq c_k(F)$. It remains to establish strict monotonicity when $r > c_k(F)$.

To do this, suppose $r_2 > r_1 > c_k(F)$. We will show that $A_k(x_k(r_2)) > A_k(x_k(r_1))$.

Since $r_1 > c_k(F)$, we have $x_k(r_1) > 0$. Moreover, $\Phi_k(x_k(r_1)) > 0$ by Remark 3. Hence

$$T_{k,r_2}(x_k(r_1)) = r_2\Phi_k(x_k(r_1)) > r_1\Phi_k(x_k(r_1)) = x_k(r_1). \tag{12}$$

Define the iteration $y_0 \equiv x_k(r_1)$ and $y_{m+1} \equiv T_{k,r_2}(y_m)$. By (12), $y_1 > y_0$. Since T_{k,r_2} is increasing, applying it preserves this inequality at every step, so (y_m) is non-decreasing. It is bounded above by r_2, so it converges to some limit L. By continuity,

$$T_{k,r_2}(L) = T_{k,r_2}\left(\lim_{m\to\infty} y_m\right) = \lim_{m\to\infty} T_{k,r_2}(y_m) = \lim_{m\to\infty} y_{m+1} = L.$$

Thus L is a fixed point of T_{k,r_2}. Because $y_m \geq y_1$ for every $m \geq 1$, the strict gap in (12) survives in the limit: $L \geq y_1 > x_k(r_1)$. Since $x_k(r_2)$ is defined as the largest fixed point of T_{k,r_2}, $x_k(r_2) \geq L > x_k(r_1)$. By Remark 3, strict monotonicity of A_k therefore gives $A_k(x_k(r_2)) > A_k(x_k(r_1))$. By Theorem 1, $a_k(r_2; F) > a_k(r_1; F)$.

(ii) Monotonicity in k. Fix the interaction rate r. By Remark 3, $\Phi_{k+1}(x) < \Phi_k(x)$ for every $x > 0$. Therefore $T_{k+1,r}(x) \leq T_{k,r}(x)$ for all $x \geq 0$.

There are two cases to consider. First, if $x_{k+1}(r) = 0$, then $x_{k+1}(r) \leq x_k(r)$ trivially. Second, suppose $x_{k+1}(r) > 0$. Then

$$T_{k,r}(x_{k+1}(r)) \geq T_{k+1,r}(x_{k+1}(r)) = x_{k+1}(r).$$

Iterating $T_{k,r}$ from $x_{k+1}(r)$ therefore produces a non-decreasing sequence bounded above by r. By the same continuity argument as in part (i), it converges to a fixed point of $T_{k,r}$ which is at least $x_{k+1}(r)$. Since $x_k(r)$ is defined as the largest fixed point of $T_{k,r}$, $x_k(r) \geq x_{k+1}(r)$.

Again, by Theorem 1,

$$\begin{aligned} a_{k+1}(r; F) &= \mathbb{E}[\psi_{k+1}(W x_{k+1}(r))] \\ &\leq \mathbb{E}[\psi_k(W x_{k+1}(r))] \\ &\leq \mathbb{E}[\psi_k(W x_k(r))] \\ &= a_k(r; F). \end{aligned}$$

We now consider the two cases in the statement of the theorem. If $r \leq c_k(F)$, then Theorem 1 gives $a_k(r; F) = 0$. Since participation is non-negative, the inequality above implies $a_{k+1}(r; F) = a_k(r; F) = 0$.

Now suppose $r > c_k(F)$. By Theorem 1, $a_k(r; F) > 0$. If $a_{k+1}(r; F) = 0$, then $a_{k+1}(r; F) < a_k(r; F)$ immediately. If $a_{k+1}(r; F) > 0$, then necessarily $x_{k+1}(r) > 0$ because $A_{k+1}(0) = 0$. By Remark 3, the first inequality in the sequence of displayed equations above is then strict. Hence $a_{k+1}(r; F) < a_k(r; F)$ in both cases. □

A.3 Proof of Proposition 1 (Jump size)

Proof. By Assumption 2, g has a unique minimizer $x_k^\star(F)$ and is strictly increasing on $[x_k^\star(F), \infty)$. For $r > c_k(F) = g(x_k^\star(F))$, the unique solution of $r = g(x)$ on this increasing branch is $x_k(r)$: the equations $r = g(x)$ and $x = r\Phi_k(x)$ are equivalent, and this branch contains the largest solution.

The map Φ_k is continuous. Indeed, if $x_n \to x$, then $W\psi_{k-1}(Wx_n) \to W\psi_{k-1}(Wx)$ almost surely, and $0 \le W\psi_{k-1}(Wx_n) \le W$. Since $\mathbb{E}[W] = 1$, dominated convergence gives $\Phi_k(x_n) \to \Phi_k(x)$. Because $\Phi_k(x) > 0$ for $x > 0$, $g(x) = x/\Phi_k(x)$ is therefore continuous on $(0, \infty)$. Its inverse on $[x_k^\star(F), \infty)$ is continuous, so $x_k(r) \downarrow x_k^\star(F)$ as $r \downarrow c_k(F)$.

Likewise, A_k is continuous by dominated convergence: $\psi_k(Wx_n) \to \psi_k(Wx)$ almost surely, and the integrands are bounded by one. Therefore, since $a_k(r; F) = A_k(x_k(r))$,

$$J_k(F) = \lim_{r \downarrow c_k(F)} a_k(r; F) = A_k(x_k^\star(F)) = \mathbb{E}[\psi_k(W\, x_k^\star(F))]\,.$$

Finally, $x_k^\star(F) > 0$, so Remark 3 gives $A_k(x_k^\star(F)) > 0$. Hence $J_k(F) > 0$. $\square$

A.4 Proof of Theorem 3 (Homogeneity maximizes the jump)

Proof of part (i): Homogeneous jump. When $W \equiv 1$, $\Phi_k(x) = \psi_{k-1}(x)$, $A_k(x) = \psi_k(x)$, and $x^\star \equiv x_k^\star(\delta_1)$ denotes the unique minimizer. It satisfies the first-order condition $\psi_{k-1}(x^\star) = x^\star\, \psi'_{k-1}(x^\star)$. Using the standard Poisson identities $\psi'_m(x) = p_{m-1}(x)$, where $p_m(x) \equiv \mathbb{P}(\text{Poisson}(x) = m)$, and $\psi_{k-1} = \psi_k + p_{k-1}$, we obtain

$$\begin{aligned} J_k^{ER} &= \psi_k(x^\star) = \psi_{k-1}(x^\star) - p_{k-1}(x^\star) = x^\star\, p_{k-2}(x^\star) - p_{k-1}(x^\star) \\ &= (k-1)\, p_{k-1}(x^\star) - p_{k-1}(x^\star) = (k-2)\, p_{k-1}(x^\star), \end{aligned}$$

where the penultimate step uses the Poisson recursion $x\, p_{k-2}(x) = (k-1)\, p_{k-1}(x)$. For $k = 3$, $x^\star$ solves $e^x = 1 + x + x^2$, yielding $J_3(\delta_1) = p_2(x^\star) \approx 0.268$, and $c_3 = \frac{x^\star}{\psi_2(x^\star)} \approx 3.35$, as stated in the theorem. $\square$

Throughout the remainder of this proof, fix $k \ge 3$ and define

$$H(v) \;\equiv\; (k-1)\, p_{k-1}(v) - \psi_{k-1}(v), \tag{13}$$

and let $\bar{x}_k = x_k^\star(\delta_1)$. By the first-order condition, $H(\bar{x}_k) = 0$; Lemma 1 below shows that this is the unique positive zero of H.

Part (ii): homogeneity maximizes the jump. The argument has two main ingredients. First, the first-order condition at the unique minimizer $x_k^\star(F)$ of $x/\Phi_k(x)$ allows us to express the problem in terms of the random variable $V = Wx_k^\star(F)$. We show that $\mathbb{E}[VH(V)] = 0$, while $A_k(x_k^\star(F)) = \mathbb{E}[\psi_k(V)] = J_k(F)$. Second, we construct a pointwise upper bound for $\psi_k(v)$ consisting of its value at the homogeneous profile $\psi_k(\bar{x}_k)$ plus a multiple of $vH(v)$. The latter term has zero expectation by the first-order condition, so taking expectations yields the desired comparison. The two lemmas below establish the properties needed to construct this pointwise bound; the bound itself is stated in Proposition 6.

Note that the second part of the lemma below applies to an arbitrary interior critical point. In this proof, we only apply it at the unique minimizer of g, but we use the more general result in the proof of Theorem 4.

Lemma 1 (Properties of H and the substitution $V = Wx^\circ$)**.**

(i) For every $v > 0$, $H'(v) = (k-2-v)\,p_{k-2}(v)$. Hence H is strictly increasing on $(0, k-2)$ and strictly decreasing on $[k-2, \infty)$, with $H(0^+) = 0$ and $H(v) \to -1$ as $v \to \infty$. In particular $\bar{x}_k > k-2$, and $H > 0$ on $(0, \bar{x}_k)$, $H < 0$ on $(\bar{x}_k, \infty)$. Moreover $|H| \le k-1$.

(ii) Let $x^\circ > 0$ be any interior critical point of $g_F(x) \equiv g(x) = x/\Phi_k(x)$, and set $V \equiv Wx^\circ$. Then

$$\mathbb{E}[V\,H(V)] = 0, \qquad \text{and} \qquad A_k(x^\circ) = \mathbb{E}[\psi_k(V)]\,. \tag{14}$$

Proof. (i) As $v \downarrow 0$, both $p_{k-1}(v)$ and $\psi_{k-1}(v)$ converge to zero, so $H(0^+) = 0$. As $v \to \infty$, $p_{k-1}(v) \to 0$ and $\psi_{k-1}(v) \to 1$, so $H(v) \to -1$.

Differentiating (13) using $\psi'_{k-1} = p_{k-2}$ and $p'_{k-1}(v) = p_{k-2}(v) - p_{k-1}(v)$ gives $H'(v) = (k-1)p_{k-2}(v) - (k-1)p_{k-1}(v) - p_{k-2}(v) = (k-2)p_{k-2}(v) - (k-1)p_{k-1}(v)$. Applying the Poisson recursion $v\,p_{k-2}(v) = (k-1)\,p_{k-1}(v)$ then gives $H'(v) = (k-2-v)\,p_{k-2}(v)$. Since $p_{k-2}(v) > 0$ for $v > 0$, H is strictly increasing on $(0, k-2)$ and strictly decreasing on $[k-2, \infty)$. It therefore rises from zero to $H(k-2) > 0$ and then falls continuously to -1, so it has exactly one positive zero. Since $H(\bar{x}_k) = 0$, that zero is $\bar{x}_k > k-2$; hence $H > 0$ on $(0, \bar{x}_k)$ and $H < 0$ on $(\bar{x}_k, \infty)$. Finally, the bound $|H| \le k-1$ holds because $0 \le \psi_{k-1} \le 1$ and $0 \le (k-1)p_{k-1} \le k-1$.

(ii) The first-order condition for a critical point x° of g_F is $\Phi_k(x^\circ) = x^\circ\,\Phi'_k(x^\circ)$, with $\Phi'_k(x) = \mathbb{E}[W^2 p_{k-2}(Wx)]$ (differentiation under the expectation is justified by dominated convergence under Assumption 1). By the definitions of V and Φ_k, $x^\circ\Phi_k(x^\circ) = \mathbb{E}[V\psi_{k-1}(V)]$. The derivative formula similarly gives $(x^\circ)^2\Phi'_k(x^\circ) = \mathbb{E}[V^2 p_{k-2}(V)]$.

Multiplying the first-order condition by x° therefore gives the first equality below. Applying the Poisson recursion $vp_{k-2}(v) = (k-1)p_{k-1}(v)$, as used in part (i), gives the second:

$$\mathbb{E}[V\,\psi_{k-1}(V)] = \mathbb{E}\big[V^2 p_{k-2}(V)\big] = (k-1)\,\mathbb{E}[V\,p_{k-1}(V)]\,,$$

which rearranges to $\mathbb{E}[VH(V)] = 0$. Integrability holds since $|VH(V)| \le (k-1)V$ and $\mathbb{E}[V] = x^\circ < \infty$. The second identity in (14) is simply the definition of A_k. $\square$

We next establish a single-crossing property that will imply that the gap in the proposed pointwise bound attains its unique maximum at $\bar{x}_k$.

Lemma 2 (A single-crossing property). *Define $f(v) \equiv v\,H(v)$. Then*

$$f'(v) \;=\; (k-1)^2\,p_{k-1}(v) \;-\; k(k-1)\,p_k(v) \;-\; \psi_{k-1}(v), \tag{15}$$

and for every constant $\epsilon \ge 0$, the function $v \mapsto f'(v) + \epsilon\,p_{k-1}(v)$ has exactly one zero on $(0,\infty)$, and is positive before it and negative after it. In particular, taking $\epsilon = 0$, f is strictly increasing on $(0, \eta_k)$ and strictly decreasing on (η_k, ∞), where η_k denotes the unique positive zero of f'. Moreover, $f'(\bar{x}_k) < 0$, and therefore $0 < \eta_k < \bar{x}_k$.

Proof. First, we compute the derivative in (15): $f' = H + vH'$, and by Lemma 1(i), $vH'(v) = (k-2-v)\,v\,p_{k-2}(v) = (k-2)(k-1)\,p_{k-1}(v) - k(k-1)\,p_k(v)$, using the recursions $v\,p_{k-2}(v) = (k-1)\,p_{k-1}(v)$ and $v\,p_{k-1}(v) = k\,p_k(v)$. Adding $H = (k-1)p_{k-1} - \psi_{k-1}$ gives (15).

For $v > 0$, dividing $f'(v) + \epsilon p_{k-1}(v)$ by $p_{k-1}(v)$ and using $p_k(v)/p_{k-1}(v) = v/k$ gives

$$\begin{aligned}\frac{f'(v) + \epsilon p_{k-1}(v)}{p_{k-1}(v)} &= (k-1)^2 + \epsilon - (k-1)\,v \\ &\quad - \sum_{i\ge 0} \frac{(k-1)!}{(k-1+i)!}\,v^i.\end{aligned} \tag{16}$$

The series (which has all positive coefficients) is obtained by dividing each term of $\psi_{k-1}(v) = \sum_{j\ge k-1} e^{-v} v^j/j!$ by $p_{k-1}(v) = e^{-v}v^{k-1}/(k-1)!$. It follows that the right-hand side of (16) is strictly decreasing in v. As $v \downarrow 0$, the series converges to 1, so the ratio converges to $(k-1)^2+\epsilon-1$. This limit is strictly positive because $k \ge 3$ and $\epsilon \ge 0$. As $v \to \infty$, the ratio tends to $-\infty$. Hence the ratio, and therefore $f'(v) + \epsilon p_{k-1}(v)$, has exactly one positive zero and is positive before it and negative after it. Finally, since $H(\bar{x}_k) = 0$ and $\bar{x}_k > k-2$, Lemma 1(i) gives $f'(\bar{x}_k) = \bar{x}_k H'(\bar{x}_k) < 0$. The unique zero η_k of f' therefore satisfies $0 < \eta_k < \bar{x}_k$. $\square$

We now introduce the pointwise bound on $\psi_k(v)$.

Proposition 6 (The pointwise bound)**.** *For all $v \geq 0$,*

$$\psi_k(v) \leq \psi_k(\bar{x}_k) + \lambda_k v H(v),$$

where

$$\lambda_k \;\equiv\; \frac{p_{k-1}(\bar{x}_k)}{\bar{x}_k\, H'(\bar{x}_k)} \;=\; -\frac{1}{(k-1)\big(\bar{x}_k - (k-2)\big)} < 0.$$

Equality holds if and only if $v = \bar{x}_k$.

Proof. Define the difference between $\psi_k(v)$ and the proposed bound by

$$G_k(v) \;\equiv\; \psi_k(v) - \psi_k(\bar{x}_k) - \lambda_k\, v\, H(v). \tag{17}$$

It suffices to show that $G_k(v) \leq 0$, with equality if and only if $v = \bar{x}_k$.

By Lemma 1(i), $H'(\bar{x}_k) = -\big(\bar{x}_k - (k-2)\big)\, p_{k-2}(\bar{x}_k)$. Combining this with the recursion $\bar{x}_k\, p_{k-2}(\bar{x}_k) = (k-1)\, p_{k-1}(\bar{x}_k)$ gives the closed form for λ_k. In particular, $\lambda_k < 0$.

We have

$$G_k'(v) = p_{k-1}(v) - \lambda_k\, f'(v) = -\lambda_k\big(f'(v) + \epsilon_k p_{k-1}(v)\big),$$
$$\epsilon_k \equiv -\frac{1}{\lambda_k} = (k-1)\big(\bar{x}_k - (k-2)\big) > 0.$$

By Lemma 2, $f'(v) + \epsilon_k p_{k-1}(v)$ has a unique positive zero, is positive before it, and is negative after it. Since $-\lambda_k > 0$, the same holds for $G_k'(v)$. That zero is $\bar{x}_k$. Indeed, by the definition $f(v) = vH(v)$ and the product rule, $f'(\bar{x}_k) = H(\bar{x}_k) + \bar{x}_k H'(\bar{x}_k) = \bar{x}_k H'(\bar{x}_k)$, where the last equality uses $H(\bar{x}_k) = 0$. Thus $\lambda_k f'(\bar{x}_k) = p_{k-1}(\bar{x}_k)$ and $G_k'(\bar{x}_k) = 0$. Hence G_k is strictly increasing on $(0, \bar{x}_k)$ and strictly decreasing on $(\bar{x}_k, \infty)$. Since its unique maximum is $G_k(\bar{x}_k) = 0$, it follows that $G_k(v) < 0$ for every $v \neq \bar{x}_k$. $\square$

We may now prove part (ii) of the theorem.

Proof of part (ii). Set $V = Wx_k^\star(F)$. By Proposition 1,

$$J_k(F) = A_k(x_k^\star(F)) = \mathbb{E}[\psi_k(V)].$$

Since $x_k^\star(F)$ is an interior critical point of g_F, Lemma 1 gives

$$\mathbb{E}[VH(V)] = 0.$$

By Proposition 6,

$$\psi_k(V) \le \psi_k(\bar{x}_k) + \lambda_k VH(V) \quad \text{almost surely.}$$

Taking expectations therefore gives

$$J_k(F) \le \psi_k(\bar{x}_k) = J_k(\delta_1).$$

Equality requires $V = \bar{x}_k$ almost surely. Hence W is constant, and the normalization $\mathbb{E}[W] = 1$ implies $W \equiv 1$, or $F = \delta_1$. The converse is immediate. Finally, note that the argument above also gives the identity

$$J_k(\delta_1) - J_k(F) = \mathbb{E}[-G_k(V)] \ge 0. \tag{18}$$

□

A.5 Proof of Theorem 4 (Jump comparison)

Fix $k \ge 3$ and interaction profiles $F_1 \preceq_* F_2$, and write $x_1^\star = x_k^\star(F_1)$. Let x_2° be any interior critical point of g_{F_2}. We compare $A_{k,F_2}(x_2^\circ)$ with $J_k(F_1)$ by evaluating the two interaction profiles at the same quantiles. Taking $x_2^\circ = x_k^\star(F_2)$ then gives $J_k(F_2) \le J_k(F_1)$, with equality if and only if $F_2 = F_1$.

We write $f(v) \equiv vH(v)$ and $\eta_k \in (0, \bar{x}_k)$, as in Lemma 2.

Recall that $F_1 \preceq_* F_2$ if and only if there exists a nondecreasing function $\rho : (0,1) \to [0,\infty)$ such that

$$Q_2(u) \;=\; \rho(u)\, Q_1(u) \qquad \text{for almost every } u \in (0,1). \tag{19}$$

Here Q_1 and Q_2 are the quantile functions of the two interaction profiles.

We rewrite $A_{k,F_2}(x_2^\circ)$ and $J_k(F_1)$ as expectations of functions of quantiles. This allows us to use the relation between Q_1 and Q_2 in (19) to compare them. Let $U \sim \text{Uniform}[0,1]$. By the standard quantile representation, $Q_i(U)$ has distribution F_i. Thus, if $W_i \sim F_i$, then

$$\mathbb{E}[h(Q_i(U))] = \mathbb{E}[h(W_i)]$$

for every measurable function h for which these expectations exist. We therefore represent the scaled interaction weights xW as

$$X \equiv x_1^\star\, Q_1(U), \qquad Y \equiv x_2^\circ\, Q_2(U).$$

Letting $\bar\rho(u) \equiv \left(x_2^\circ/x_1^\star\right)\rho(u)$, we have $Y = \bar\rho(U)X$ almost surely. Since $x_2^\circ/x_1^\star > 0$ and ρ is nondecreasing, $\bar\rho$ is also nondecreasing.

Note that we may normalize $\bar\rho = 0$ wherever $Q_1 = 0$. By (19), $Q_2 = 0$ almost everywhere on this set, so the normalization leaves $Y = \bar\rho(U)X$ unchanged almost surely. Moreover, since Q_1 is nonnegative and nondecreasing, it follows that if $Q_1(u) = 0$, then $Q_1(v) = 0$ for every $0 < v < u$. Setting $\bar\rho$ to zero on these quantiles therefore preserves its monotonicity.

The first-order conditions for g_{F_1} at $x_1^\star$ and for g_{F_2} at x_2° give, by Lemma 1(ii),

$$\mathbb{E}[f(X)] = \mathbb{E}[f(Y)] = 0. \tag{20}$$

The expectations are finite because $|f(v)| \le (k-1)v$ and $\mathbb{E}[Q_i(U)] = \mathbb{E}[W_i] = 1$. In particular,

$$\mathbb{E}[X] = x_1^\star \mathbb{E}[Q_1(U)] = x_1^\star, \qquad \mathbb{E}[Y] = x_2^\circ \mathbb{E}[Q_2(U)] = x_2^\circ.$$

The definition of A_k and Proposition 1 also give

$$A_{k,F_2}(x_2^\circ) = \mathbb{E}[\psi_k(Y)], \qquad J_k(F_1) = \mathbb{E}[\psi_k(X)].$$

The next lemma chooses a rescaling of X so that $f(tX)$ bounds $f(Y)$ at almost every quantile.

Lemma 3 (Rescaling X)**.** *There exist $0 < t \leq 1$ and $u_0 \in [0,1)$ such that, almost everywhere,*

$$u < u_0 \implies Y \leq tX \leq \eta_k, \qquad \textit{and} \qquad u > u_0 \implies Y \geq tX \geq \eta_k.$$

For this choice of t, $f(Y) \leq f(tX)$ almost everywhere.

Proof. First, note that $\bar\rho X$ is nondecreasing, since it is a product of nonnegative nondecreasing functions. It is also almost everywhere equal to Y. Let $u_0 \equiv \sup\{u : \bar\rho(u)X(u) < \eta_k\}$, with $u_0 = 0$ if the set is empty. By monotonicity, $\bar\rho(u)X(u) < \eta_k$ for all $u < u_0$ and $\bar\rho(u)X(u) \geq \eta_k$ for all $u > u_0$. The value of $\bar\rho X$ at u_0 does not affect any of the comparisons below because they apply almost everywhere.

Observe that $u_0 = 1$ is impossible. In that case, $0 \leq Y < \eta_k$ almost everywhere, and $\mathbb{E}[Y] = x_2^\circ > 0$ implies $\mathbb{P}(Y > 0) > 0$. Since $f(0) = 0$ and $f > 0$ on $(0, \eta_k]$, this would imply $\mathbb{E}[f(Y)] > 0$, contradicting (20).

To choose t, we use the limiting values of X and $\bar\rho$ on either side of u_0. Write

$$x_- \equiv \sup_{u<u_0} X(u), \qquad x_+ \equiv \inf_{u>u_0} X(u),$$
$$\bar\rho_- \equiv \sup_{u<u_0} \bar\rho(u), \qquad \bar\rho_+ \equiv \inf_{u>u_0} \bar\rho(u).$$

If $u_0 = 0$, the two suprema are over empty sets; in this case we set $x_- = \bar\rho_- = 0$. Monotonicity gives $x_- \leq x_+$ and $\bar\rho_- \leq \bar\rho_+$. Taking one-sided limits of $\bar\rho X$ on either side of u_0 gives

$$\bar\rho_-\, x_- \leq \eta_k \leq \bar\rho_+\, x_+. \tag{21}$$

The first inequality in (21) implies $\bar\rho_- \leq \eta_k/x_-$, and the second implies $\eta_k/x_+ \leq \bar\rho_+$. Here we take $\eta_k/0 = +\infty$, since x_- can be zero. Together with $\bar\rho_- \leq \bar\rho_+$ and $x_- \leq x_+$, these two inequalities imply that the intervals $[\bar\rho_-, \bar\rho_+]$ and $[\eta_k/x_+,\ \eta_k/x_-]$ intersect.

Choose t in the intersection of $[\bar\rho_-, \bar\rho_+]$ and $[\eta_k/x_+,\ \eta_k/x_-]$. Since $u_0 < 1$ and X

and $\bar{\rho}$ are finite and nondecreasing on $(0,1)$, both x_+ and $\bar{\rho}_+$ are finite. The inequality $\bar{\rho}_+ x_+ \geq \eta_k > 0$ also implies $x_+ > 0$. Thus any such choice of t is finite and strictly positive, since $0 < \eta_k / x_+ \leq t \leq \bar{\rho}_+$.

We now show that for any such choice of t, $f(Y) \leq f(tX)$ almost everywhere. For $u < u_0$, we have $\bar{\rho}(u) \leq \bar{\rho}_- \leq t$ and $X(u) \leq x_-$, so almost everywhere $Y = \bar{\rho} X \leq tX \leq t\,x_- \leq \eta_k$. When $u > u_0$, we have $\bar{\rho}(u) \geq \bar{\rho}_+ \geq t$ and $X(u) \geq x_+$, so almost everywhere $Y \geq tX \geq t\,x_+ \geq \eta_k$. By Lemma 2, f is strictly increasing on $[0, \eta_k]$ and strictly decreasing on $[\eta_k, \infty)$. For $u < u_0$, the first property gives $f(Y) \leq f(tX)$, since $Y \leq tX \leq \eta_k$. For $u > u_0$, the second gives $f(Y) \leq f(tX)$, since $Y \geq tX \geq \eta_k$. Thus $f(Y) \leq f(tX)$ almost everywhere, as claimed.

It remains to show that any choice of t above satisfies $t \leq 1$. Recall $H(v) = (k-1)\,p_{k-1}(v) - \psi_{k-1}(v)$. Since $v\,p_{k-2}(v) = (k-1)\,p_{k-1}(v)$ and $\Phi'_{k,F_1}(x) = \mathbb{E}[W_1^2 p_{k-2}(W_1 x)]$ (Lemma 1(ii)), we have

$$\mathbb{E}[W_1 H(W_1 x)] = x\,\Phi'_{k,F_1}(x) - \Phi_{k,F_1}(x).$$

Since $g'_{F_1}(x) = -\mathbb{E}[W_1 H(W_1 x)]/\Phi_{k,F_1}(x)^2$, Assumption 2, which requires $g'_{F_1}(x) > 0$ for all $x > x_1^\star$, gives

$$\mathbb{E}[W_1 H(W_1 x)] < 0 \qquad \text{for all } x > x_1^\star. \tag{22}$$

The inequality $f(Y) \leq f(tX)$ established above and (20) imply

$$0 = \mathbb{E}[f(Y)] \;\leq\; \mathbb{E}[f(tX)] = t\,x_1^\star\,\mathbb{E}[W_1 H(W_1 t\,x_1^\star)].$$

If $t > 1$, then $t\,x_1^\star > x_1^\star$ and (22) makes the right-hand side strictly negative—a contradiction. Hence $t \leq 1$.

□

Proof of the theorem. Take t and u_0 from Lemma 3. We consider the cases $t = 1$ and $t < 1$. In both cases we show that $\mathbb{E}[\psi_k(Y)] \leq \mathbb{E}[\psi_k(X)]$, with equality only if $X = Y$ almost everywhere.

Case 1: $t = 1$. In this case, $f(Y) \leq f(X)$ almost everywhere with $\mathbb{E}[f(Y)] = \mathbb{E}[f(X)] = 0$, so $f(Y) = f(X)$ almost everywhere. Next, recall that when $u < u_0$, both X and Y lie in $[0, \eta_k]$, where f is strictly increasing. Hence $X = Y$ almost everywhere on $\{u < u_0\}$. Likewise, when $u > u_0$, both lie in $[\eta_k, \infty)$, where f is strictly decreasing, so $X = Y$ almost everywhere on $\{u > u_0\}$. Hence $X = Y$ almost everywhere, which gives $\mathbb{E}[\psi_k(Y)] = \mathbb{E}[\psi_k(X)]$.

Case 2: $t < 1$. Here we divide the proof into two further subcases: whether $\mathbb{P}(Y > X) = 0$ or $\mathbb{P}(Y > X) > 0$.

Suppose $\mathbb{P}(Y > X) = 0$. Then $Y \leq X$ almost everywhere. Hence $\mathbb{E}[\psi_k(Y)] \leq \mathbb{E}[\psi_k(X)]$ since ψ_k is strictly increasing, with equality if and only if $X = Y$ almost everywhere.

Now suppose that $\mathbb{P}(Y > X) > 0$. Let $u_c \equiv \sup\{u : \bar{\rho}(u) < 1\}$. By monotonicity, $\bar{\rho}(u) < 1$ for $u < u_c$ and $\bar{\rho}(u) \geq 1$ for $u > u_c$. Since $Y = \bar{\rho}X$ almost everywhere, the event $Y > X$ requires $X > 0$ and $\bar{\rho}(U) > 1$. Since this event has positive probability, $\bar{\rho}$ exceeds 1 at some quantile $u < 1$. By monotonicity it remains above 1 at every higher quantile, so $u_c < 1$.

Moreover, $u_c \geq u_0$. To see this, recall that the construction of t gives $\bar{\rho}(u) \leq \bar{\rho}_- \leq t < 1$ for every $u < u_0$. Thus $\bar{\rho}$ cannot reach 1 below u_0, so $u_c \geq u_0$.

Set $c \equiv \inf_{u > u_c} X(u)$. Since every $u > u_c$ is also greater than u_0, Lemma 3 gives $tX \geq \eta_k$ almost everywhere on this interval. Since X is nondecreasing, its limit from above at u_c satisfies

$$c \;\geq\; \eta_k/t \;>\; \eta_k.$$

By monotonicity and the definition of u_c, we have

$$u < u_c \implies Y \leq X \leq c, \qquad u > u_c \implies Y \geq X \geq c, \tag{23}$$

almost everywhere. As in the proof of Proposition 6, we subtract a multiple of f from ψ_k to obtain a function that is increasing up to a chosen point and decreasing thereafter. Here the chosen point is c. Since $c > \eta_k$, $f'(c) < 0$, and we define

$$Q_c(v) \equiv \psi_k(v) - \lambda_c\, f(v), \qquad \lambda_c \equiv \frac{p_{k-1}(c)}{f'(c)} < 0.$$

Then $Q_c'(v) = -\lambda_c\big(f'(v) + \epsilon_c\, p_{k-1}(v)\big)$ with $\epsilon_c \equiv -1/\lambda_c > 0$, and $Q_c'(c) = 0$ by construction. By Lemma 2, c is the *unique* positive zero of Q_c', so Q_c is strictly increasing on $[0, c]$ and strictly decreasing on $[c, \infty)$. For $u < u_c$, (23) gives $Y \leq X \leq c$. Since Q_c is increasing on $[0, c]$, we have $Q_c(Y) \leq Q_c(X)$. For $u > u_c$, (23) gives $Y \geq X \geq c$. Q_c is decreasing on $[c, \infty)$, so again we have $Q_c(Y) \leq Q_c(X)$. Thus $Q_c(Y) \leq Q_c(X)$ almost everywhere. The random variables $Q_c(X)$ and $Q_c(Y)$ are integrable since $|Q_c(v)| \le 1 + |\lambda_c|\,(k-1)\,v$ and $\mathbb{E}[X], \mathbb{E}[Y] < \infty$. Taking expectations and using $\mathbb{E}[f(X)] = \mathbb{E}[f(Y)] = 0$ as in (20), we have

$$\mathbb{E}[\psi_k(Y)] \;\le\; \mathbb{E}[\psi_k(X)],$$

with equality if and only if $Q_c(Y) = Q_c(X)$ almost everywhere. Since $X, Y \in [0, c]$ for $u < u_c$ and Q_c is strictly increasing on this interval, $X = Y$ almost everywhere there. Applying the same argument for $u > u_c$, where $X, Y \in [c, \infty)$ and Q_c is strictly decreasing, shows that $X = Y$ almost everywhere. Hence $\mathbb{P}(Y > X) > 0$ implies the inequality is strict.

Thus, at any interior critical point x_2° of g_{F_2}, we have

$$A_{k,F_2}(x_2^\circ) = \mathbb{E}[\psi_k(Y)] \le \mathbb{E}[\psi_k(X)] = J_k(F_1).$$

In particular, taking $x_2^\circ = x_k^\star(F_2)$ gives $J_k(F_2) \le J_k(F_1)$. Equality requires $X = Y$ almost everywhere. Since $\mathbb{E}[X] = x_1^\star$ and $\mathbb{E}[Y] = x_2^\circ$, this implies $x_2^\circ = x_1^\star$. The equality $X = Y$ is therefore equivalent to $x_1^\star\, Q_1(u) = x_1^\star\, Q_2(u)$ almost everywhere. Dividing both sides by $x_1^\star > 0$ gives $Q_1 = Q_2$ almost everywhere, hence $F_1 = F_2$. Conversely, if $F_1 = F_2$, equality is immediate. $\square$

A.6 Proof of Proposition 2 (Cutoff comparison)

Proof. Fix $x > 0$ and let $h_x(w) \equiv w\,\psi_{k-1}(wx)$. Then $h_x(0) = 0$ and $h_x(w)/w = \psi_{k-1}(wx)$ is nondecreasing in w, so h_x is starshaped. Since $0 \le h_x(w) \le w$ and $\mathbb{E}[W_i] = 1$, the expectations $\mathbb{E}[h_x(W_i)]$ are finite. Hence if $F_1 \le_{\text{ss}} F_2$, then $\Phi_{k,F_1}(x) = \mathbb{E}[h_x(W_1)] \le \mathbb{E}[h_x(W_2)] = \Phi_{k,F_2}(x)$ for every $x > 0$. Since $\Phi_{k,F_i}(x) > 0$ for all $x > 0$, it follows that $g_{F_1}(x) \ge g_{F_2}(x)$ for all $x > 0$. Taking infima gives $c_k(F_1) \ge c_k(F_2)$. $\square$

A.7 Proof of Proposition 3 ($c_k(F)$ strictly increasing in k)

Proof. By Remark 3, $\Phi_{k+1}(x) < \Phi_k(x)$ for every $x > 0$. Hence $x/\Phi_{k+1}(x) > x/\Phi_k(x)$ for every $x > 0$. Hence at the minimizer $x_{k+1}^\star$ of $g_{k+1}(x) = x/\Phi_{k+1}(x)$, we have

$$c_{k+1}(F) \;=\; \frac{x_{k+1}^\star}{\Phi_{k+1}(x_{k+1}^\star)} \;>\; \frac{x_{k+1}^\star}{\Phi_k(x_{k+1}^\star)} \;\ge\; \inf_{x>0} \frac{x}{\Phi_k(x)} \;=\; c_k(F). \qquad \square$$

A.8 Proof of Remark 2 (Bernoulli dilution scaling)

Proof. We first establish the scaling identities for Φ_k, A_k, and the critical cutoff. We then show that dilution preserves Assumption 2, which allows us to obtain the scaling identity for the jump.

Let $W \sim F$ and $B \sim \text{Bernoulli}(\nu)$ be independent and define $W^{(\nu)} \equiv (B/\nu)W$.

Part (i): scaling identities. Since $B \in \{0, 1\}$, we may write

$$\Phi_k^{(\nu)}(x) = \mathbb{E}\left[\frac{B}{\nu}W\,\psi_{k-1}\left(\frac{B}{\nu}Wx\right)\right] = \nu \cdot \mathbb{E}\left[\frac{1}{\nu}W\,\psi_{k-1}\left(\frac{1}{\nu}Wx\right)\right] = \Phi_k(x/\nu).$$

Also $\psi_k(0) = 0$ for $k \geq 1$, so

$$A_k^{(\nu)}(x) = \mathbb{E}\left[\psi_k\left(\frac{B}{\nu}Wx\right)\right] = \nu\,\mathbb{E}\left[\psi_k\left(\frac{1}{\nu}Wx\right)\right] = \nu\,A_k(x/\nu).$$

Thus $g^{(\nu)}(x) = x/\Phi_k^{(\nu)}(x) = \nu g(x/\nu)$ and taking infima yields $c_k(F^{(\nu)}) = \nu c_k(F)$.

Part (ii): preservation of Assumption 2 and jump scaling. Suppose F satisfies Assumption 2. By Part (i), $g^{(\nu)}(x) = \nu g(x/\nu)$, so $g^{(\nu)}$ has a unique minimizer at $\nu x_k^\star(F)$. Differentiating,

$$(g^{(\nu)})'(x) = \nu \cdot g'(x/\nu) \cdot \frac{1}{\nu} = g'(x/\nu).$$

For $x > \nu x_k^\star(F)$, $x/\nu > x_k^\star(F)$, so $g'(x/\nu) > 0$ because F satisfies Assumption 2. Hence $(g^{(\nu)})'(x) > 0$ for all $x > \nu x_k^\star(F)$, and therefore $F^{(\nu)}$ also satisfies Assumption 2. The jump scaling identity now follows:

$$J_k(F^{(\nu)}) = A_k^{(\nu)}(\nu x_k^\star(F)) = \nu A_k(x_k^\star(F)) = \nu J_k(F). \qquad \square$$

A.9 Proof of Proposition 4 (The missing middle)

Proof. Let $c \equiv c_k(F)$ and $J \equiv J_k(F) > 0$, and write

$$a(r) \equiv a_k(r; F), \qquad u(r) \equiv \alpha r - \frac{1}{2}r^2, \qquad \pi_\beta(r; F) \equiv u(r) - \beta a(r).$$

For $r \leq c$, Theorem 1 gives $a(r) = 0$, so $\pi_\beta(r; F) = u(r)$. Since $\alpha > c$ by assumption, $u'(r) = \alpha - r > 0$ on $[0, c]$. Thus $\pi_\beta(r; F) < u(c) = \pi_\beta(c; F)$ for all $r < c$. In Lemma 9 of Online Appendix OA.12.6, we show that there exists a unique $\bar{\beta} \in (0, \infty)$ such that

$$\sup_{s>c} \pi_{\bar{\beta}}(s; F) = u(c).$$

When $\beta < \bar{\beta}$, the supremum is strictly larger than $u(c)$, and when $\beta > \bar{\beta}$ it is strictly smaller.

Proof of part (i). If $\beta > \bar{\beta}$, then for every $r > c$, $\pi_\beta(r; F) \le \sup_{s>c} \pi_\beta(s; F) < u(c) = \pi_\beta(c; F)$. Since we also have $\pi_\beta(r; F) < \pi_\beta(c; F)$ for all $r < c$, the unique maximum is obtained at $r = c$.

Proof of part (ii). If $\beta < \bar{\beta}$, then $\sup_{s>c} \pi_\beta(s; F) > u(c) = \pi_\beta(c; F)$, so c cannot be a maximizer. As shown above, the maximizer cannot lie strictly below c either. Hence every optimal $r^\star$ satisfies $r^\star > c$.

Now fix any optimal $r^\star > c$. Since $r^\star$ is optimal and $\beta < \bar{\beta}$, $\pi_\beta(r^\star; F) > \pi_\beta(c; F)$, that is, $\alpha r^\star - \frac{1}{2}(r^\star)^2 - \beta a(r^\star) > \alpha c - \frac{1}{2}c^2$. In particular, since participation must be at least as large as the jump ($a(r^\star) \ge J$), a necessary condition for $\pi_\beta(r^\star; F) \ge \pi_\beta(c; F)$ is

$$\alpha r^\star - \tfrac{1}{2}(r^\star)^2 - \beta J \ge \alpha c - \tfrac{1}{2}c^2.$$

Writing $r^\star = c + \varepsilon$ with $\varepsilon > 0$ gives $\frac{1}{2}\varepsilon^2 - (\alpha - c)\varepsilon + \beta J \le 0$. Hence ε must be at least as large as the smallest root of this quadratic, $\varepsilon_- = (\alpha - c) - \sqrt{(\alpha - c)^2 - 2\beta J}$. Finally, since $\sqrt{x^2 - y} \le x - y/(2x)$ for $x > 0$, we obtain $\varepsilon_- \ge \beta J/(\alpha - c)$. Therefore $r^\star \ge c + \beta J_k(F)/(\alpha - c) = c_k(F) + d_k(F)$, as required. $\square$

A.10 Proof of Proposition 5 (Comparative statics and tipping points)

Proof. Throughout, write $c \equiv c_k(F)$ and $a(r) \equiv a_k(r; F)$.

Part (i): Monotonicity. Fix $0 \le \beta_1 < \beta_2$ and choose $r_i \in R(\beta_i)$. Optimality implies

$$\pi_{\beta_1}(r_1; F) \ge \pi_{\beta_1}(r_2; F), \qquad \pi_{\beta_2}(r_2; F) \ge \pi_{\beta_2}(r_1; F).$$

Adding these inequalities and canceling the common terms $\alpha r - \frac{1}{2}r^2$ gives $(\beta_2 - \beta_1)(a(r_2) - a(r_1)) \le 0$, so $a(r_2) \le a(r_1)$.

As argued in the proof of Proposition 4, for any β, an interaction rate strictly smaller than c can never be optimal. Hence $r_1, r_2 \ge c$. If $r_2 = c$, then $r_1 \ge r_2$

automatically.

Suppose $r_2 > c$. Then Theorem 1(i) gives $a(r_2) > 0$. Since $a(r_2) \leq a(r_1)$, we also have $a(r_1) > 0$, so again by Theorem 1(i) we must have $r_1 > c$. On (c, ∞), the map $a(\cdot)$ is strictly increasing by Theorem 2(i). Therefore $a(r_2) \leq a(r_1)$ implies $r_2 \leq r_1$.

Part (ii): Strict monotonicity. In Lemma 10 of Online Appendix OA.12.7, we show that a is C^2 on (c, ∞) with $a' > 0$. Assume $0 \leq \beta_1 < \beta_2 < \bar{\beta}$ and that each $R(\beta_i)$ is the singleton $\{r_i\}$. By Proposition 4(ii), both optimizers satisfy $r_1, r_2 > c$, and by Part (i), $r_2 \leq r_1$.

Suppose for a contradiction that $r_2 = r_1 \equiv \hat{r}$. Since $\hat{r} \in (c, \infty)$ is an interior point, it must satisfy the first-order condition for both π_{β_1} and π_{β_2}. That is,

$$0 = \frac{\partial}{\partial r} \pi_\beta(\hat{r}; F) = \alpha - \hat{r} - \beta a'(\hat{r}).$$

Since $a'(\hat{r}) > 0$, this equation determines β uniquely at a given $\hat{r}$, which forces $\beta_1 = \beta_2$, a contradiction. Hence $r_2 < r_1$, and because $a(\cdot)$ is strictly increasing on (c, ∞), $a(r_2) < a(r_1)$.

Part (iii): Local derivative. Let $\beta < \bar{\beta}(\alpha; F)$ be such that the maximizer is unique, interior, and nondegenerate, and denote it by $r^\star(\beta) > c$. Here interior means $r^\star(\beta) \in (c, \infty)$, and nondegeneracy means that the second derivative of the payoff at the maximizer is nonzero; since this is a maximum, $\frac{\partial^2}{\partial r^2} \pi_\beta(r^\star(\beta); F) < 0$. Define

$$L(\beta, r) \equiv \frac{\partial}{\partial r} \pi_\beta(r; F) = \alpha - r - \beta a'(r).$$

Then

$$L(\beta, r^\star(\beta)) = 0, \qquad \frac{\partial L}{\partial r}(\beta, r^\star(\beta)) = -1 - \beta a''(r^\star(\beta)) = \frac{\partial^2}{\partial r^2} \pi_\beta(r^\star(\beta); F) < 0.$$

By the implicit function theorem, $r^\star(\cdot)$ is differentiable at β, with

$$\frac{\mathrm{d}r^\star}{\mathrm{d}\beta} = -\frac{\partial L/\partial \beta}{\partial L/\partial r} = -\frac{-a'(r^\star)}{-1 - \beta a''(r^\star)} = -\frac{a'(r^\star)}{1 + \beta a''(r^\star)} < 0,$$

where the final inequality uses $a'(r^\star) > 0$ and $-1 - \beta a''(r^\star) < 0$. Writing $\bar{a}^\star = a(r^\star(\beta))$, we then have

$$\frac{\mathrm{d}\bar{a}^\star}{\mathrm{d}\beta} = a'(r^\star) \frac{\mathrm{d}r^\star}{\mathrm{d}\beta} < 0.$$

□

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Online Appendix

"Controlling Complex Contagions"

Alastair Langtry DJ Thornton

This appendix collects supplementary material: microfoundations for equilibrium selection and the payoff specification, robustness exercises and examples, and technical lemmas used to support some of the main-text proofs.

OA.1 Micro-founding the largest equilibrium

In Section 3 we focus on the largest equilibrium of the second stage (i.e. the one with the most participating agents). As discussed there, there are two natural interpretations of this focus, depending on the application. In entrenched-behavior settings, best-response dynamics from the all-participate profile select the largest equilibrium as a literal prediction. In coordination settings, the largest equilibrium characterizes the coordination capacity that the principal designs against.

Here, we formalize these two interpretations by providing two micro-foundations for the focus on the largest equilibrium. In the first, agents are boundedly rational and play myopic best responses. In the second, agents are fully rational and reach equilibrium using only local information and a short communication phase. Both micro-foundations yield the same outcome: the set of participating agents is the k-core of the realized graph.

Unsophisticated agents: myopic best-response dynamics

Set-up. Suppose that the game is played over $n+1$ periods. At $t = 1$, the principal chooses $r \geq 0$, and Nature realizes the graph G (as in our baseline model). Additionally, at $t = 1$, all agents participate *non-strategically* (i.e. $a_{i,t=1} = 1$ for all i). In each subsequent period $t \geq 2$, all agents best respond myopically to actions in the preceding period $t - 1$. Agent i's action in period $t \geq 2$ is thus determined by the best response

$$BR^{(n)}_{i,t}(M_{i,t-1}) = \begin{cases} 1, & \text{if } M_{i,t-1} \geq k, \\ 0, & \text{if } M_{i,t-1} < k, \end{cases} \tag{24}$$

where $M_{i,t-1} = \sum_{j \neq i} G_{ij} a_{j,t-1}$. For convenience, we write $a_i \equiv a_{i,n+1}$ for i's action in the final period and assume that the principal cares about participation at the end of the game, not during it.

The result. Exactly as in our baseline model, the unique limit of the above dynamic process is that each agent participates if and only if he belongs to the k-core of the graph G. In other words, Remark 1 holds unchanged.

Discussion. In addition to myopia, this set-up assumes that all agents start off by participating. From a technical standpoint, this guarantees convergence of the best-response dynamics to the largest equilibrium of the static game. The logic is very similar to that laid out after Remark 1: agents iteratively drop out when they have fewer than k participating neighbors, and the process terminates at the k-core. The only difference is that agents are updating myopically rather than performing analogous reasoning in their heads.

Economically, this set-up imposes $a = 1$ as the pre-existing default. The myopic-updating view therefore best fits settings where the principal wants to eradicate or reduce pre-existing behaviors. This is natural when considering the adoption of new technologies or new social norms: the old versions were the status quo before the start of the game, so everyone must have been using the old technology or adhering to the old norm. Note that in this interpretation our terminology becomes less clean: "participate" means to use the *old* technology or norm, and "abstain" means to adopt the *new* one.

Sophisticated agents: local communication

For this microfoundation, we make the additional assumption that an agent's payoff from abstaining does not depend on how many of her neighbors participate, and that her payoff from participating is weakly increasing in the number of participating neighbors. This ensures that an agent can never strictly benefit by getting her neighbors to abstain.

At $t = 1$, the principal chooses $r \geq 0$, Nature realizes the graph G, and all agents announce $m_{i,1} \equiv \text{P}$. This is followed by a "communication phase" of n periods (periods 2 through $n + 1$) and an "action phase" at $t = n + 2$. In each period $t \geq 2$, agents observe only (i) how many neighbors they have and (ii) what each of those neighbors announced in the previous period.

Communication phase (periods 2 to n+1). During the communication phase, every agent i announces a *cheap talk* message $m_{i,t} \in \{P, A\}$, interpreted as "I plan to Participate" or "I will Abstain." Suppose agents adopt the rule:

$$m_{i,t} = \begin{cases} \text{A} & \text{if strictly fewer than } k \text{ neighbors sent P at } t-1, \\ \text{P} & \text{otherwise.} \end{cases}$$

Action phase (period n+2). Consider the strategy s_i given by: play $a_i = 1$ if and only if at least k of i's neighbors sent P in the final communication round (period $n+1$).

The result. The message rule implements the usual k-core peeling process: after (at most) n periods, no further agents switch to sending A's and the only agents who still announce P are the vertices in the k-core of G.

Given the strategy s_i for the action phase, any agent who still announces P in the final communication round has at least k neighbors who also announced P. These agents will therefore choose $a_i = 1$ in the action phase. Hence playing $a_i = 1$ is a best response for every agent who announces P at time $n+1$, that is, for every agent in the k-core. Under the stipulated rule, an agent who sees less than k neighbors send P in the final round will play $a_i = 0$. This is a best response, since less than k of his neighbors will participate. Thus every agent's action under the action-phase rule is a best response to other agents' actions when all agents follow the given communication procedure.

In the communication phase, no agent can profit by unilaterally deviating from the message rule. To see this, first consider an agent i who would eventually send A under the given rule. We show that there is no alternative sequence of messages by i that can induce $\geq k$ of her neighbors to participate in the action phase, and therefore there is no incentive to unilaterally deviate from A to P. It suffices to prove this claim when the agent deviates to sending P in every period, since this is the strongest support she could possibly give to her neighbors—every other agent's message under the rule, and others' final action, is weakly increasing in the number of neighbors they have who send P.

Let t be the first round in which the rule tells i to send A. Then it must be that strictly less than k of i's neighbors sent P at $t-1$. If i continues sending P, the peeling process can only reduce the number of i's neighbors who continue to send P, and therefore the final number of i's neighbors sending P must be $< k$. Therefore, the number of i's neighbors who participate in the action phase is also $< k$, whence i prefers not to participate either. Therefore even sending P in every period cannot generate enough support to make participation worthwhile for i, and so no deviation to P in any subset of periods can generate enough support to make participation worthwhile either. Moreover, since i's payoff from abstaining is independent of her neighbors' participation, no such deviation can improve her payoff from abstaining either. In sum, if the rule ever tells the agent to send A, deviating from its prescription can never strictly increase the payoff from i's optimal strategy, whether she chooses $a_i = 0$ or $a_i = 1$.

If instead the rule never tells the agent to send A (that is, it prescribes P in every period), then deviating to A can only reduce final participation and therefore the agent's final payoff from participating. By assumption, such a deviation also cannot affect the agent's payoff from abstention, so the agent has no strict incentive to deviate. Therefore the prescribed message profile $m = (m_{i,t})_{i \in N, 1 \leq t \leq n+1}$ is itself incentive-compatible.

Consequently, (m, s) is a Nash equilibrium of the extensive-form game. The agents who play $a_i = 1$ in the final period are exactly those who belong to the k-core of G.

Discussion. Agents here are fully rational and use the communication phase to perform the same iterated elimination that the "unsophisticated" agents achieve through myopic play. Crucially, each agent needs only *local* information: his own degree and the most recent messages of his neighbors. The procedure demonstrates that the k-core is a natural characterization of equilibrium even when agents are fully sophisticated, as in our baseline model (see Remark 1).

OA.2 Micro-founding the principal's benefit from the interaction rate

In Section 3 we assumed that the principal receives per-capita benefits of αr when she chooses an interaction rate r. Here, we open up that black box, and outline a game where agents choose bilateral actions with each of their friends.

The game. As in our baseline model, the principal chooses $r \geq 0$ at $t = 1$, and then Nature realizes the graph G (as described in Section 3). At $t = 2$, each agent simultaneously chooses whether or not to take a binary, and *neighbor-specific* action, $b_{ij} \in \{0, 1\}$ with each of his neighbors. This is in addition to the action $a_i \in \{0, 1\}$ present in the baseline model. Agents have preferences represented by a utility function $U_i = u_i + v_i$, where u_i is as in Equation (2) (from the baseline model) and v_i is:

$$v_i = \sum_{j \in \mathcal{N}_i} (\gamma b_{ij} + \delta b_{ij} b_{ji}), \tag{25}$$

where $\gamma > 0, \delta > 0$ and $\mathcal{N}_i := \{j : G_{ij}^{(n)} = 1\}$ is the set of i's neighbors. The principal then receives a benefit proportional to $\frac{1}{n} \sum_i \sum_j b_{ij}$—the per-capita number of neighbor-specific actions.

Discussion. Once agents have formed links (formally, after Nature has drawn the network), agents are choosing whether or not to "engage" with each of their neighbors in some way. For employees of a firm, this engagement may be conversations about work-relevant topics. For citizens, this may be conversations about job openings, product recommendations, or invitations to community events. For users of social media platforms, engagement may be interacting with a neighbor's post. But note that the precise form of engagement could differ across pairs of agents. For example, an employee of a firm could discuss different work-related topics with each of his neighbors.

We view these choices to engage as initiating relatively "low-stakes" interactions, and as ones where an agent can receive *some* benefit even if engagement is not reciprocated. Reciprocated interaction simply provides a higher benefit. Additionally, this engagement also provides some spillover benefits to the principal. For example, work-relevant conversations by employees help spread best practice within a firm—something the firm benefits from. Greater sharing of job openings or product recommendations by citizens can improve the functioning of labor and product markets—something that can benefit the government. And a social media platform sells more advertising when there is more engagement by users.

Outcomes. It is clear from the set-up that all agents i have a strictly dominant strategy for the neighbor-specific actions b_{ij}: choose $b_{ij} = 1$ for all $j \in \mathcal{N}_i$. So the principal's benefit $\frac{1}{n}\sum_i \sum_j b_{ij}$ is equal to $\frac{1}{n}\sum_i |\mathcal{N}_i|$, which is the average degree in the network ($|\mathcal{N}_i|$ is the number of neighbors i has—i.e., his degree). By a standard Law of Large Numbers, the average degree converges to the expected degree—which is equal to r by assumption. Each agent's expected degree is $r \times W_i$, and the W_i's are i.i.d. draws from a distribution F with $\mathbb{E}[W] = 1$. In short, we have $\lim_{n\to\infty} \frac{1}{n}\sum_i \sum_j b^*_{ij} = r$.

OA.3 Heavy-tail edge case (outside Assumption 1)

Lemma 4 is exactly where the finite-variance restriction is used to force $g(x) = x/\Phi_k(x) \to \infty$ as $x \downarrow 0$. If one leaves this moment class (for example, sufficiently heavy-tailed profiles with $\mathbb{E}[W^2] = \infty$), that boundary behavior can fail. Then the minimizer of g need not be interior, and participation can emerge continuously with $J_k(F) = 0$. All results in the main text take Assumption 1 as given.

OA.4 Convex order does not imply monotonicity of the jump

The following example shows that ordering interaction profiles by the convex order does not guarantee monotonicity of the jump.

Remark 4 (Convex order counterexample for $k = 3$)**.** *Define two interaction profiles, each with* $\mathbb{E}[W] = 1$*:*

$$F_1: \quad W = \begin{cases} \frac{30}{49} & w.p.\ 0.7, \\ \frac{60}{49} & w.p.\ 0.1, \\ \frac{110}{49} & w.p.\ 0.2; \end{cases} \qquad\qquad F_2: \quad W = \begin{cases} \frac{10}{81} & w.p.\ 0.5, \\ \frac{130}{81} & w.p.\ 0.4, \\ \frac{240}{81} & w.p.\ 0.1. \end{cases}$$

One can verify that $F_1 \leq_{\text{cx}} F_2$ (i.e., F_2 is a mean-preserving spread of F_1) by checking *the stop-loss order:* $\mathbb{E}_{F_1}[(W-t)^+] \le \mathbb{E}_{F_2}[(W-t)^+]$ *for all* $t \ge 0$.[38] *The variances are* $\mathrm{Var}(W) \approx 0.420$ *under* F_1 *and* $\mathrm{Var}(W) \approx 0.916$ *under* F_2. *However, direct computation gives*

$$J_3(F_1) \approx 0.1003 \quad < \quad J_3(F_2) \approx 0.1040.$$

The more dispersed profile has the larger *jump.*

The obstruction is that $J_k(F) = A_k(x_k^\star(F))$ *depends on both the argmin* $x_k^\star(F)$ *of* $g^F(x) = x/\Phi_k^F(x)$ *and the composition with* A_k^F. *Neither the argmin map nor the composition preserves convex-order monotonicity. The pair above is in fact star-incomparable: on the common refinement of the two profiles' probability intervals (masses* 0.5, 0.2, 0.1, 0.1, 0.1*), the quantile ratios of* F_2 *to* F_1 *are*

$$0.202,\ 2.621,\ 1.311,\ 0.715,\ 1.320,$$

which is not monotone. As Theorem 4 shows, the convex order is the wrong order for ranking the jump; the star order is the right one.

OA.5 Robustness to a positive-degree periphery

This section shows that the comparative-statics results derived for the two-class family F_ν (Section 5) are robust to giving the periphery strictly positive interaction weights.

A perturbed class of core-periphery networks. For $\epsilon \in [0, 1)$, define

$$F_{\nu,\epsilon}: \quad W = \begin{cases} \epsilon & \text{w.p. } 1-\nu, \\ w_c(\nu,\epsilon) \equiv \dfrac{1-\epsilon(1-\nu)}{\nu} & \text{w.p. } \nu, \end{cases}$$

so that $\mathbb{E}[W] = 1$. Equivalently, we may write $W = \epsilon + (1-\epsilon)X$ where $X \sim F_\nu$. One can think of W as a compound distribution: first, every agent receives a weight ϵ, then the remaining mass is distributed according to the core-periphery class studied in the main text. When $\epsilon = 0$, we recover exactly that class: $F_{\nu,0} = F_\nu$. When $\epsilon > 0$,

[38] Both distributions have $\mathbb{E}[W] = 1$. Direct computation of $\mathbb{E}[(W-t)^+]$ for each distribution confirms the inequality for all $t \ge 0$, which is equivalent to the convex order.

all agents have strictly positive expected degree.

Key objects. Writing $\Phi_{k,\epsilon}(x) \equiv \mathbb{E}_{F_{\nu,\epsilon}}[W\psi_{k-1}(Wx)]$, $A_{k,\epsilon}(x) \equiv \mathbb{E}_{F_{\nu,\epsilon}}[\psi_k(Wx)]$, and $g_\epsilon(x) \equiv x/\Phi_{k,\epsilon}(x)$, the critical cutoff is $c_k(F_{\nu,\epsilon}) = \inf_{x>0} g_\epsilon(x)$ and the jump is $J_k(F_{\nu,\epsilon}) = A_{k,\epsilon}(x^\star_\epsilon)$, where $x^\star_\epsilon$ is the minimizer of g_ϵ.

Proposition 7 (Positive-degree periphery robustness)**.** *Fix $k \geq 3$ and $\nu \in (0,1)$. Assume $g_0(x) = x/\Phi_{k,0}(x)$ has a unique minimizer $x^\star_0$ (i.e., Assumption 2 holds for F_ν). For the approximation bounds in part (ii), assume in addition that $g_0''(x^\star_0) > 0$. Then:*

(i) (Continuity.) *$c_k(F_{\nu,\epsilon}) \to \nu\, c_k^{ER}$ and $J_k(F_{\nu,\epsilon}) \to \nu\, J_k^{ER}$ as $\epsilon \downarrow 0$.*

(ii) (Approximation bounds.) *The critical cutoff satisfies*

$$c_k(F_{\nu,\epsilon}) = \nu\, c_k^{ER} + O(\epsilon).$$

The jump satisfies the corresponding bound:

$$J_k(F_{\nu,\epsilon}) = \nu\, J_k^{ER} + O(\epsilon),$$

where the implicit constants in $O(\epsilon)$ depend only on k and ν.

Proof. The argument proceeds in three steps.

Step 1: Uniform convergence on compact sets. Since ψ_m is C^∞ and 1-Lipschitz, and the support of $F_{\nu,\epsilon}$ is bounded uniformly in ϵ (for fixed ν), $\Phi_{k,\epsilon}$ and $A_{k,\epsilon}$ converge uniformly to $\Phi_{k,0}$ and $A_{k,0}$ on every compact interval $[0,b]$, hence

$$\sup_{x\in[0,b]} \left|\Phi_{k,\epsilon}(x) - \Phi_{k,0}(x)\right| = O(\epsilon), \qquad \sup_{x\in[0,b]} \left|A_{k,\epsilon}(x) - A_{k,0}(x)\right| = O(\epsilon).$$

To obtain these two bounds, compare the periphery and core contributions with their values at $\epsilon = 0$. The periphery contributes zero at $\epsilon = 0$. For positive ϵ, its contribution to $\Phi_{k,\epsilon}$ is at most $(1-\nu)\epsilon$, since $\psi_{k-1} \leq 1$, and its contribution to $A_{k,\epsilon}$ is at most $(1-\nu)b\epsilon$ on $[0,b]$, since $\psi_k(0) = 0$ and ψ_k is 1-Lipschitz.

Under the positive-degree periphery distribution, the weight placed on the core falls from $1/\nu$ to $w_c = 1/\nu - (1-\nu)\epsilon/\nu$. The reduction in this weight is $(1-\nu)\epsilon/\nu$. Because ψ_{k-1} and ψ_k are 1-Lipschitz, replacing their argument x/ν by $w_c x$ changes either tail probability by at most $x(1-\nu)\epsilon/\nu$. For $x \in [0,b]$, this is at most $b(1-\nu)\epsilon/\nu$, giving a single bound valid throughout the interval. The core contribution to $A_{k,\epsilon}$ is $\nu\psi_k(w_c x)$, so its change is bounded by this estimate times ν. For $\Phi_{k,\epsilon}$, the core contribution is $\nu w_c \psi_{k-1}(w_c x)$: the coefficient νw_c also falls by $(1-\nu)\epsilon$, and $\psi_{k-1} \le 1$ bounds the effect of that change. Thus both core contributions change by at most a constant times ϵ on $[0,b]$.

Differentiating the expectation formulas for $\Phi_{k,\epsilon}$ and $A_{k,\epsilon}$ under the two-point distribution $F_{\nu,\epsilon}$ with respect to x gives the same $O(\epsilon)$ bounds for their first derivatives, because the derivatives of ψ_m are bounded on the relevant compact interval.

Step 2: Minimizers lie in a common compact interval. The Poisson tail $\psi_{k-1}(z)$ is $O(z^{k-1})$ as $z \downarrow 0$: its power series starts at degree $k-1$. Since $W \le 1/\nu$ for every $\epsilon \in [0,1)$, it follows that $\Phi_{k,\epsilon}(x) \le Cx^{k-1}$ for all sufficiently small x, with the same constant C for every ϵ. Thus $g_\epsilon(x) \ge C^{-1}x^{2-k} \to \infty$ as $x \downarrow 0$. At the other end, $\Phi_{k,\epsilon}(x) \le \mathbb{E}[W] = 1$ implies $g_\epsilon(x) \ge x \to \infty$. Note that both lower bounds do not depend on ϵ. By Step 1, we have $g_\epsilon(x_0^\star) \to g_0(x_0^\star)$, so the value of g_ϵ at this fixed point is bounded for small ϵ. We can therefore choose $0 < a < x_0^\star < b$ so that the lower bounds outside $[a,b]$ exceed this value for every sufficiently small ϵ. For fixed k and ν, every minimizer of g_ϵ across the family $F_{\nu,\epsilon}$ with sufficiently small ϵ must therefore lie in this same interval $[a,b]$.

Step 3: Convergence and approximation bounds. On $[a,b]$, $\Phi_{k,0}$ is bounded away from zero, so Step 1 also gives $\sup_{x\in[a,b]} |g_\epsilon(x) - g_0(x)| = O(\epsilon)$. By Step 2 and the uniqueness of $x_0^\star$, the maximum theorem (applied to $-g_\epsilon$ on $[a,b]$) gives $x_\epsilon^\star \to x_0^\star$ and $c_k(F_{\nu,\epsilon}) = g_\epsilon(x_\epsilon^\star) \to g_0(x_0^\star) = \nu\, c_k^{ER}$. The difference between the minimum values of two functions is no larger than their largest absolute difference on the common domain $[a,b]$. The $O(\epsilon)$ bound just established therefore gives the claimed approximation bound for the critical cutoff.

For the jump, decompose

$$J_k(F_{\nu,\epsilon}) - \nu J_k^{ER} = \left[A_{k,\epsilon}(x_\epsilon^\star) - A_{k,0}(x_\epsilon^\star)\right] + \left[A_{k,0}(x_\epsilon^\star) - A_{k,0}(x_0^\star)\right].$$

The first bracket is $O(\epsilon)$ by Step 1 (uniform convergence of $A_{k,\epsilon}$ to $A_{k,0}$ on $[a,b]$). The second bracket tends to zero because $x_\epsilon^\star \to x_0^\star$ and $A_{k,0}$ is continuous. This proves the jump convergence in part (i).

For the stronger bound in part (ii), use $g_0''(x_0^\star) > 0$. The two-point formulas make

$g'_\epsilon(x)$ continuously differentiable in x and ϵ near $(x_0^\star, 0)$. Since $g'_0(x_0^\star) = 0$ and its derivative with respect to x is strictly positive there, the implicit function theorem gives a unique solution to $g'_\epsilon(x) = 0$ near $x_0^\star$ that varies smoothly with ϵ. The minimizers converge to $x_0^\star$ and hence coincide with this solution for small ϵ. Smooth dependence on ϵ gives $x_\epsilon^\star - x_0^\star = O(\epsilon)$. Because $A'_{k,0}$ is bounded on $[a, b]$, the second bracket is also $O(\epsilon)$. Hence

$$J_k(F_{\nu,\epsilon}) = \nu\, J_k^{ER} + O(\epsilon).$$

□

OA.6 The principal's optimal interaction rate is non-monotone in k

The discussion in Section 6 ("Raising the threshold") noted that the principal's optimal interaction rate r^* can move in either direction as the participation threshold k rises. The mechanism is a regime transition. For any (α, β, F) with $\alpha > c_k(F)$, the principal's optimum sits either at the cutoff itself or strictly above it (Proposition 4). When $\alpha \leq c_k(F)$, the principal simply goes to the unconstrained naive optimum $r^{\text{naive}} = \alpha$: there is no need to consider participation. As k rises, the cutoff $c_k(F)$ rises; in the examples below, the jump $J_k(F)$ also rises. These changes can shift the principal between regimes, and the new r^* may lie above or below the previous one, depending on which transition is triggered. The two examples below illustrate. Both fix $\alpha = 5$ and $\beta = 1.8$.

Example 1: r^* rises with k. Take the homogeneous benchmark, $W \equiv 1$:

k	c_k	J_k	regime	r^*	$\bar{a}^*$
3	3.35	0.27	suppress at cutoff	3.35	0
4	5.15	0.44	naive	5.00	0
5	6.80	0.54	naive	5.00	0

At $k = 3$, the cutoff lies below the principal's naive optimum, and she pins r at the cutoff to preclude participation while keeping aggregate interaction as high as possible. As k rises to 4, c_k moves above α; the principal can reach the naive optimum without inducing any participation. So she does exactly that, and chooses $r^{\text{naive}} = \alpha = 5$. The transition raises r^* from 3.35 to 5.

Example 2: r^* falls with k. Take the Bernoulli dilution with $\nu = 0.3$: $W = 1/\nu$ with probability ν and $W = 0$ otherwise.

k	c_k	J_k	regime	r^*	$\bar{a}^*$
3	1.01	0.080	interior	5.00	0.30
8	3.40	0.206	interior	4.99	0.30
10	4.26	0.221	suppress at cutoff	4.26	0

For $k \leq 8$, the principal sits in the interior regime: she sets r just below the naive optimum α, accepts participation $\bar{a}^* \approx \nu = 0.3$ (the saturating share of the active core), and pays the participation cost $\beta \bar{a}_k^*$. As k rises to 10, the jump J_k becomes large enough to make participation undesirable at any level, and the principal optimally chooses an interaction rate at the new cutoff $c_{10} = 4.26$. The transition lowers r^* by roughly 0.7.[39]

Together, these examples confirm the claim made in Section 6: the comparative static of r^* in k depends on which regime transition the change triggers. A transition out of suppression-at-cutoff into the naive regime raises r^*, while a transition in the other direction lowers it.

OA.7 The jump need not be increasing in k

The qualification in Section 5.3 is necessary: the jump can fall as the participation threshold rises, even when Assumption 2 holds at both thresholds. Consider the mean-one profile

$$W = \begin{cases} 5/7 & \text{w.p. } 4/5, \\ 15/7 & \text{w.p. } 1/5. \end{cases}$$

Set $t = 5x/7$. Then

$$g_k(7t/5) = \frac{49}{5} \frac{t}{4\psi_{k-1}(t) + 3\psi_{k-1}(3t)}, \qquad A_k(7t/5) = \frac{4}{5}\psi_k(t) + \frac{1}{5}\psi_k(3t).$$

[39] Continuing this profile to higher k recovers Example 1's mechanism: at $k = 15$, $c_{15} \approx 6.30 > \alpha$, the cutoff is out of reach, and r^* rises back to $r^{\text{naive}} = \alpha = 5$. So within this single profile, r^* falls and then rises as k grows; the comparative static of r^* in k is non-monotone in both directions.

Writing $p_j(z) = e^{-z}z^j/j!$, the derivative of the first expression has the sign of

$$Q_k(t) = 4\big[\psi_{k-1}(t) - tp_{k-2}(t)\big] + 3\big[\psi_{k-1}(3t) - 3tp_{k-2}(3t)\big].$$

For $a = k - 2$, differentiation gives

$$Q_k'(t) = p_a(t)R_a(t), \qquad R_a(t) = 4(t-a) + 3^{a+2}e^{-2t}(3t-a).$$

For $a = 4, 5$, R_a has exactly one positive zero. Indeed, it is negative on $t \le a/3$, positive on $t \ge a$, and on $(a/3, a)$ its sign is the sign of

$$\ell_a(t) = \log\left(\frac{3^{a+2}(3t-a)}{4(a-t)}\right) - 2t.$$

The derivative of ℓ_a has sign pattern $+, -, +$, and at its local minimum $v_a = (2a + \sqrt{a(a-3)})/3$,

$$\ell_4(v_4) \approx 0.736, \qquad \ell_5(v_5) \approx 0.119.$$

Thus Q_k first decreases and then increases. Since $Q_k(0) = 0$ and $Q_k(t) \to 7$, it has a unique positive zero for $k = 6, 7$; hence Assumption 2 holds in both cases. Direct computation gives

$$t_6^\star \approx 2.458886, \quad J_6(F) \approx 0.180424,$$
$$t_7^\star \approx 2.903740, \quad J_7(F) \approx 0.176237.$$

Therefore $J_7(F) < J_6(F)$: raising the participation threshold can strictly reduce the size of the jump. The displayed sign checks can also be certified using elementary bounds; we report their numerical values for brevity.

OA.8 The critical cutoff can rise under a mean-preserving spread

This section gives two examples in which a mean-preserving spread raises the critical cutoff.

A rare-hub spread that raises c_3. Let $W \in \{\frac{1470}{1499}, 30\}$ with $\mathbb{P}(W = 30) = 1/1500$ (so $\mathbb{E}[W] = 1$; $\mathrm{Var}(W) \approx 0.58$). Direct computation gives $c_3(F) = 3.355535 > c_3(\delta_1) = 3.350919$, with a unique global minimizer of g at $x^\star = 1.7372$ and $g' > 0$

beyond it, so Assumption 2 holds. (The hub branch of g has a local dip near $x \approx 0.074$ with value $4.785 > c_3(F)$, which Assumption 2 permits: it constrains g only beyond the minimizer.) The jump is $J_3(F) = 0.2442 < J_3(\delta_1)$, as Theorem 3 requires.

The mechanism is instructive. A rare hub of total weight $s = \varepsilon M$ adds approximately s to Φ_k (its ψ-factor saturates) but approximately nothing to Φ'_k; funding it costs the bulk s units of weight near $w \approx 1$, each worth $\frac{d}{dw}\big[w\,\psi_{k-1}(w\bar{x}_k)\big] = \psi_{k-1}(\bar{x}_k) + \bar{x}_k\,p_{k-2}(\bar{x}_k) = 2\,\psi_{k-1}(\bar{x}_k)$ at the benchmark first-order condition. Since $2\psi_{k-1}(\bar{x}_k) > 1$, the trade raises the cutoff. The hub must not be self-sustaining on its own: if it were, it would seed its own core at small x and the cutoff would instead collapse (the multi-jump case of Online Appendix OA.12.2).

The direction flips with the threshold. For $W = 1 + Z$ with $\mathbb{E}[Z] = 0$ and Z small, a second-order expansion gives

$$\Phi_k(x) - \psi_{k-1}(x) \;\approx\; \tfrac{1}{2}\,\mathrm{Var}(W)\,x\,p_{k-2}(x)\,(k - x),$$

so by the envelope theorem the variance-response of c_k at δ_1 has the sign of $-(k - \bar{x}_k)$. Numerically $\bar{x}_k < k$ for $k = 3, 4, 5$ ($\bar{x}_5 = 4.88$) and $\bar{x}_k > k$ from $k = 6$ ($\bar{x}_6 = 6.32$): small heterogeneity lowers the cutoff at low thresholds and raises it at high ones. The flip is visible non-locally within the gamma family: c_6, c_8, c_{10}, c_{20} initially rise with dispersion while c_3, c_4, c_5 fall (c_{10}: from 14.192 at the homogeneous limit to 14.596 at shape $a = 20$)—and the jump falls at every k, again as Theorem 3 requires.

OA.9 The two orders are incomparable

This section certifies the claims in the main text about the star order $\leq_*$ and the starshaped order $\leq_{\mathrm{ss}}$: neither implies the other, and each ranks exactly one of the two summary statistics. Throughout, profiles have mean one, and $T_F(t) \equiv \mathbb{E}_F[W\,\mathbf{1}(W > t)]$ denotes the weighted tail, so that $F_1 \leq_{\mathrm{ss}} F_2$ if and only if $T_{F_1} \leq T_{F_2}$ pointwise (Shaked and Shanthikumar, 2007, Theorem 4.A.54).

Strictly positive profiles are never starshaped-comparable. Let W^{s} denote the size-biased weight. Since $\mathbb{E}[W] = 1$, $T_F(t) = \mathbb{P}(W^{\mathrm{s}} > t)$, so the weighted-tail comparison above is equivalent to first-order stochastic dominance of the size-biased weights, as noted in the main text's definition of the starshaped order. Also,

$\mathbb{E}[1/W^{\mathrm{s}}] = \mathbb{E}[W \cdot (1/W)\,\mathbf{1}(W > 0)] = \mathbb{P}(W > 0)$. Suppose $F_1 \le_{\mathrm{ss}} F_2$, i.e. $W_1^{\mathrm{s}} \le_{\mathrm{st}} W_2^{\mathrm{s}}$. Applying the first-order stochastic dominance relation $W_1^{\mathrm{s}} \le_{\mathrm{st}} W_2^{\mathrm{s}}$ to the strictly increasing function $z \mapsto -1/z$ gives $\mathbb{P}(W_1 > 0) \ge \mathbb{P}(W_2 > 0)$. If both profiles are strictly positive, both sides equal one; equality of the expectation of a strictly monotone function under first-order dominance forces W_1^{s} and W_2^{s} to share a distribution, and size-biasing is invertible on $(0, \infty)$ for mean-one laws, so $F_1 = F_2$. Hence moving up the starshaped order across distinct mean-one profiles requires a strictly larger mass of zero-weight agents.

The star order does not imply the starshaped order. The mean-one gamma family is a chain in the star order (Section OA.11), yet no two of its members are starshaped-comparable, since every gamma profile is strictly positive. (Directly: the size-biased laws are $\Gamma(a + 1, \text{rate } a)$, whose survival functions cross—near zero the shape parameter dominates, in the tail the rate does.) This is how the cutoff can move non-monotonically along a star chain: at $k = 10$, the cutoff rises from 14.19 (the homogeneous limit) through 14.31 (shape 100) to 14.70 (shape 10), then falls to 14.48 (shape 5).

The starshaped order does not imply the star order. Take $W_1 \in \{0, 1, 2\}$ with probabilities $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$ and $W_2 \in \{0, 2\}$ with probabilities $(\frac{1}{2}, \frac{1}{2})$. Both have mean one and $T_{F_1} \le T_{F_2}$ pointwise, so $F_1 \le_{\mathrm{ss}} F_2$; but on the common refinement of their quantile functions the ratio takes the values 0, 2, 1, so the pair is not star-ordered.

The starshaped order does not rank the jump. Diluting the homogeneous profile ($\nu = \frac{1}{2}$) moves up the starshaped order and halves the jump: $J_3 = 0.2676 \to 0.1338$. By contrast, take $G_1 \in \{0, 1, 6\}$ with probabilities $(\frac{5}{8}, \frac{1}{4}, \frac{1}{8})$ and $G_2 \in \{0, 6\}$ with probabilities $(\frac{5}{6}, \frac{1}{6})$: then $G_1 \le_{\mathrm{ss}} G_2$ (weighted tails $\frac{3}{4} \le 1$ on $[1, 6)$, equal elsewhere), both satisfy Assumption 2, and the jump *rises*: $J_3(G_1) \approx 0.0358 < 0.0446 \approx J_3(G_2)$. So the starshaped order ranks the jump in neither direction, just as the star order does not rank the cutoff (Section OA.8).

OA.10 Near-homogeneous stability

Proof. Throughout this proof, write $g^F(x) \equiv x/\Phi_k(x)$ when we want to display the dependence on the interaction profile F, and abbreviate $g^{ER}(x) \equiv x/\psi_{k-1}(x)$ for the

Erdős–Rényi case (where $W \equiv 1$ gives $\Phi_k = \psi_{k-1}$). Fix $k \geq 3$ and let $x^\star \equiv x_k^{\star,ER}$ denote the unique minimizer of g^{ER}, with minimum value $c^{ER} \equiv c_k^{ER}$.

Proof strategy. We organize the proof in two parts, sharing common setup. *Setup* (Steps 1 and 2): Step 1 confines the minimizer of g^F to a fixed compact interval $[a, b]$ around $x^\star$ that does not depend on F, turning the rest of the argument into a uniform-approximation problem on a compact set; Step 2 shows $g^F \to g^{ER}$ uniformly on $[a, b]$ as $\sigma(F) \to 0$. *Part A* (Step 3) proves cutoff continuity, $|c_k(F) - c^{ER}| \to 0$. *Part B* invokes Lemma 8 (see Section OA.12.5 below) which verifies Assumption 2 for sufficiently homogeneous profiles, and then combines compactness with uniform approximation of A_k to prove continuity of the jump, $|J_k(F) - J_k^{ER}| \to 0$.

Step 1: compactness of the minimization problem near ER. Fix $\eta > 0$. Because $g^{ER}(x) \to \infty$ as $x \downarrow 0$ and as $x \to \infty$, there exist $0 < a < x^\star < b < \infty$ such that

$$\inf_{x \in (0,a] \cup [b,\infty)} g^{ER}(x) \;\geq\; c^{ER} + 3\eta. \tag{26}$$

We will show that for F sufficiently close to δ_1, the minimizer of g^F must lie in $[a, b]$ as well.

Fix b as above. Note that for every F, $\Phi_k(x) \leq \mathbb{E}[W] = 1$ (since $\psi_{k-1} \leq 1$), so $g^F(x) = x/\Phi_k(x) \geq x$ for all $x \geq 0$. Thus, for the chosen b, $g^F(x) \geq x \geq b$ for all $x \geq b$. In particular, if $b \geq c^{ER} + 3\eta$ (we may enlarge b to ensure this), then

$$\inf_{x \geq b} g^F(x) \;\geq\; c^{ER} + 3\eta \qquad \text{for all } F. \tag{27}$$

Next, fix $a > 0$ as in (26) and (if needed) shrink it further so that $a \leq 1/2$. By Lemma 6 (see Section OA.12.3 below), for any interaction profile F with dispersion $\sigma(F) = \sigma$ and any $x \in (0, a]$, $g^F(x) \geq 1/(4x + 6\sigma^2) \geq 1/(4a + 6\sigma^2)$. Choose $\varepsilon_1 \in (0, 1]$ small enough (and, if needed, shrink a once more) so that $1/(4a + 6\varepsilon_1^2) \geq c^{ER} + 3\eta$. Then for all F with $\sigma(F) \leq \varepsilon_1$,

$$\inf_{x \in (0,a]} g^F(x) \;\geq\; c^{ER} + 3\eta. \tag{28}$$

Combining (27) and (28), for all F with $\sigma(F) \leq \varepsilon_1$: if the global minimum of g^F is below $c^{ER} + 3\eta$, then every minimizer must lie in the compact interval $[a, b]$. We now show that the minimum is indeed achieved below this level.

Step 2: uniform closeness of g^F to g^{ER} on $[a,b]$. On $[a,b]$, ψ_{k-1} is bounded below by $m \equiv \min_{x\in[a,b]} \psi_{k-1}(x) > 0$. By Lemma 7 (see Section OA.12.4 below), $\sup_{x\in[a,b]} |\Phi_k(x) - \psi_{k-1}(x)| \to 0$ as $\sigma(F) \to 0$. In particular, there exists $\varepsilon_2 > 0$ such that $\sigma(F) \le \varepsilon_2$ implies $\sup_{x\in[a,b]} |\Phi_k(x) - \psi_{k-1}(x)| \le m/2$. By the triangle inequality, $\Phi_k(x) \ge \psi_{k-1}(x) - |\Phi_k(x) - \psi_{k-1}(x)| \ge m - m/2 = m/2$ for all $x \in [a,b]$. Therefore, for such F and all $x \in [a,b]$,

$$|g^F(x) - g^{ER}(x)| = \left| \frac{x}{\Phi_k(x)} - \frac{x}{\psi_{k-1}(x)} \right| = x \cdot \frac{|\Phi_k(x) - \psi_{k-1}(x)|}{\Phi_k(x)\psi_{k-1}(x)} \le b \cdot \frac{\sup_{[a,b]} |\Phi_k - \psi_{k-1}|}{(m/2) \cdot m}.$$

The denominator $(m/2) \cdot m$ is a fixed positive constant; the numerator $\sup_{[a,b]} |\Phi_k - \psi_{k-1}| \to 0$ as $\sigma(F) \to 0$ by Lemma 7. Hence $\sup_{x\in[a,b]} |g^F(x) - g^{ER}(x)| \to 0$ as $\sigma(F) \to 0$.

Part A (Step 3): cutoff continuity. Let $c(F) \equiv c_k(F) = \inf_{x>0} g^F(x)$. Since $x^\star \in [a,b]$, Step 2 gives

$$c(F) \le g^F(x^\star) \le g^{ER}(x^\star) + \sup_{[a,b]} |g^F - g^{ER}| = c^{ER} + o(1) \quad \text{as } \sigma(F) \to 0.$$

For $\sigma(F)$ small enough that the $o(1)$ term is below 3η (and $\sigma(F) \le \varepsilon_1$), we have $c(F) < c^{ER} + 3\eta$, so by Step 1 the infimum is achieved at some $x^F \in [a,b]$.

For the matching lower bound,

$$c^{ER} = g^{ER}(x^\star) \le g^{ER}(x^F) \le g^F(x^F) + \sup_{[a,b]} |g^F - g^{ER}| = c(F) + o(1).$$

Combining, $|c(F) - c^{ER}| \to 0$ as $\sigma(F) \to 0$, proving the cutoff continuity claim.

Part B: continuity of the jump. By Lemma 8 (see Section OA.12.5 below), for $\sigma(F)$ sufficiently small, g^F has a unique minimizer $x_k^\star(F)$ and $J_k(F) = A_k(x_k^\star(F))$ is well-defined. By Step 1, $x_k^\star(F) \in [a,b]$ for $\sigma(F)$ small.

Convergence of the minimizer. Suppose, for contradiction, that there is a sequence $\sigma(F_n) \to 0$ with $x_k^\star(F_n) \not\to x^\star$. By compactness, some subsequence $x_k^\star(F_{n_j}) \to x' \in [a,b]$ with $x' \neq x^\star$. Step 2 gives $g^{F_{n_j}}(x_k^\star(F_{n_j})) - g^{ER}(x_k^\star(F_{n_j})) \to 0$, and continuity of g^{ER} gives $g^{ER}(x_k^\star(F_{n_j})) \to g^{ER}(x')$. By Part A, $g^{F_{n_j}}(x_k^\star(F_{n_j})) = c_k(F_{n_j}) \to c^{ER}$. Combining, $g^{ER}(x') = c^{ER}$—but $x^\star$ is the unique minimizer of g^{ER}, contradiction. So $x_k^\star(F) \to x^\star$ as $\sigma(F) \to 0$.

Continuity of the jump. By Lemma 7, $\sup_{[a,b]} |A_k(x) - \psi_k(x)| \to 0$ as $\sigma(F) \to 0$. Combined with continuity of ψ_k and $x_k^\star(F) \to x^\star$,

$$|J_k(F) - J_k^{ER}| = |A_k(x_k^\star(F)) - \psi_k(x^\star)| \le \sup_{[a,b]} |A_k - \psi_k| + |\psi_k(x_k^\star(F)) - \psi_k(x^\star)| \to 0,$$

proving the claimed convergence. □

Remark 5 (Local quadratic loss)**.** *The deficit identity (equation 18 in the main appendix) also gives the local rate at which heterogeneity reduces the jump. Consider a family of profiles*

$$W = 1 + Z, \qquad \mathbb{E}[Z] = 0, \qquad \|Z\|_\infty \to 0,$$

and write

$$x_F \equiv x_k^\star(F), \qquad \Delta_F \equiv x_F - \bar{x}_k, \qquad V \equiv W x_F.$$

The representation $W = 1 + Z$ simply centers the perturbation around the homogeneous profile, and the normalization $\mathbb{E}[W] = 1$ implies $\mathbb{E}[Z] = 0$.

Step 1: the deficit is the expectation of an approximately quadratic function. *The function $-G_k$ is nonnegative and vanishes only at $\bar{x}_k$ (Proposition 6); its derivative also vanishes there ($G_k'(\bar{x}_k) = 0$, shown in the same proof). A second-order Taylor expansion around $\bar{x}_k$ therefore has no constant and no linear term:*

$$-G_k(v) \;=\; \tfrac{1}{2}\left|G_k''(\bar{x}_k)\right|(v-\bar{x}_k)^2 + o\big((v-\bar{x}_k)^2\big), \qquad G_k''(\bar{x}_k) \;=\; \frac{p_{k-2}(\bar{x}_k)\,(k-2-2\bar{x}_k)}{(k-1)\,d} \;<\; 0,$$

where the closed form follows by direct differentiation, using $H(\bar{x}_k) = 0$; numerically $G_3''(\bar{x}_3) \approx -0.487$.

Step 2: how far V sits from $\bar{x}_k$. *Decompose the deviation into the part coming from the perturbation and the part coming from the shifted minimizer:*

$$V - \bar{x}_k \;=\; x_F\, Z \;+\; \Delta_F.$$

Squaring and taking expectations, the cross term vanishes because $\mathbb{E}[Z]=0$*:*

$$\mathbb{E}\big[(V-\bar{x}_k)^2\big] \;=\; x_F^2\ \mathrm{Var}(W) \;+\; \Delta_F^2.$$

Both departures are uniformly small—$\sup|V-\bar{x}_k| \le x_F\|Z\|_\infty + |\Delta_F| \to 0$*, using* $x_F \to \bar{x}_k$ *from Section OA.10—and this uniformity is what allows the expectation to pass through the Taylor remainder:*

$$J_k(\delta_1) - J_k(F) \;=\; \mathbb{E}\big[-G_k(V)\big] \;=\; \tfrac{1}{2}\left|G_k''(\bar{x}_k)\right| \Big(x_F^2\ \mathrm{Var}(W) + \Delta_F^2\Big)\big(1+o(1)\big).$$

Step 3: the minimizer shift is second order, so only the variance term survives. *Expanding* Φ_k *in the perturbation,*

$$\mathbb{E}\big[(1+Z)\,\psi_{k-1}\big((1+Z)x\big)\big] - \psi_{k-1}(x) \;=\; \mathbb{E}[Z]\cdot\big(\psi_{k-1}(x) + x\,p_{k-2}(x)\big) \;+\; O\big(\mathbb{E}[Z^2]\big),$$

and the linear term vanishes because $\mathbb{E}[Z]=0$*: the perturbation moves* Φ_k *by* $O(\mathrm{Var}(W))$ *only. Because the benchmark minimizer is nondegenerate (*$H'(\bar{x}_k)<0$*; Lemma 1), a standard implicit-function argument then gives* $\Delta_F = O(\mathrm{Var}(W))$*, and hence* $\Delta_F^2 = o(\mathrm{Var}(W))$*. Combining the three steps, and using* $x_F^2 \to \bar{x}_k^2$*,*

$$J_k(\delta_1) - J_k(F) \;=\; \kappa_k\ \mathrm{Var}(W)\,\big(1+o(1)\big), \qquad \kappa_k \;\equiv\; \tfrac{1}{2}\left|G_k''(\bar{x}_k)\right|\bar{x}_k^2.$$

Numerically, $\kappa_3 \approx 0.782$*: near the benchmark, an interaction profile with dispersion* $\sigma(F)=0.3$ *has a jump roughly seven percentage points below the homogeneous jump* $J_3(\delta_1)\approx 0.268$*.*

OA.11 Parametric families

The two families are chains. For the mean-one lognormal profile with dispersion parameter σ, the quantile function is $Q_\sigma(u) = \exp\big(-\frac{\sigma^2}{2} + \sigma\,\Phi_N^{-1}(u)\big)$, where Φ_N^{-1} is the standard normal quantile. For $\sigma_2 > \sigma_1$,

$$\frac{Q_{\sigma_2}(u)}{Q_{\sigma_1}(u)} = \exp\Big(\tfrac{\sigma_1^2-\sigma_2^2}{2} + (\sigma_2-\sigma_1)\,\Phi_N^{-1}(u)\Big)$$

is strictly increasing in u, so $F_{\sigma_1} \preceq_* F_{\sigma_2}$. For the gamma family, van Zwet (1964, §4.3.4) shows that the family is a (maximal) chain in the convex-transform order, decreasing in the shape parameter: for shapes $a_2 < a_1$, the function $T = F_{a_2}^{-1} \circ F_{a_1}$ is convex on $[0, \infty)$ (see also Saunders and Moran, 1978; Arab and Oliveira, 2019). Since $T(0) = 0$ and T is convex, $T(x)/x$ is nondecreasing, and hence the quantile ratio $Q_{a_2}(u)/Q_{a_1}(u) = T(Q_{a_1}(u))/Q_{a_1}(u)$ is nondecreasing in u; rescaling to mean one multiplies the ratio by a constant and preserves the order. Hence $F_{a_1} \preceq_* F_{a_2}$ for $a_2 < a_1$.

Assumption 2 holds for every gamma profile. Since $x \mapsto z$ is a strictly increasing bijection, work in z. Using $\partial_z I_z(k-1, a+1) = z^{k-2} q^a / B(k-1, a+1)$ and $\frac{d}{dz} \log g = \frac{1}{zq} - \frac{\partial_z I}{I}$, the critical-point condition $\frac{d}{dz} \log g = 0$ is

$$I_z(k-1,\, a+1) \;=\; \frac{z^{k-1} q^{a+1}}{B(k-1,\, a+1)} \;=\; (k-1)\, \pi_{k-1}, \qquad \pi_j \equiv \frac{(a+1)_j}{j!}\, z^j q^{a+1},$$

where π_j is the negative-binomial mass with parameters $(a+1, q)$. Dividing through by π_{k-1}, the condition reads

$$T_a(z) \;\equiv\; \sum_{\ell \geq 0} \frac{(a+k)_\ell}{(k)_\ell}\, z^\ell \;=\; k-1. \tag{29}$$

$T_a(0) = 1 < k-1$, T_a is strictly increasing in z, and $T_a(z) \geq 1/(1-z) \to \infty$ as $z \uparrow 1$; hence (29) has a unique root $z(a) \in (0,1)$. Moreover $\operatorname{sgn}\big(\frac{d}{dz} \log g\big) = \operatorname{sgn}\big(T_a(z) - (k-1)\big)$, so g strictly falls before $z(a)$ and strictly rises after: **Assumption 2 holds for every gamma profile**.

Monotonicity and the limit. Fix shapes $a_2 < a_1$. Both profiles satisfy Assumption 2 by the previous paragraph, and $F_{a_1} \preceq_* F_{a_2}$, so Theorem 4 gives $J_k(F_{a_2}) < J_k(F_{a_1})$: the jump is strictly decreasing in dispersion along the family. Since the dispersion of the shape-a profile is $1/\sqrt{a} \to 0$ as $a \to \infty$, The near-homogeneous stability result in Section OA.10 gives $J_k(F_a) \to J_k(\delta_1)$; combined with monotonicity, $J_k(F_a) \uparrow J_k(\delta_1)$. The lognormal claims follow in the same way from the chain property above, wherever Assumption 2 is maintained.

OA.12 Supplementary proofs

This subsection collects the proofs of the technical lemmas that support—but are not particularly important in and of themselves—the proofs in Appendix A.

OA.12.1 Small-x behavior of Φ_k under finite variance

Lemma 4 (Small-x behavior of Φ_k under finite variance)**.** *Fix $k \geq 3$ and let $W \sim F$ satisfy $\mathbb{E}[W] = 1$, $\mathbb{P}(W > 0) > 0$, and $\mathbb{E}[W^2] < \infty$. Then*

$$\lim_{x \downarrow 0} \frac{\Phi_k(x)}{x} = 0.$$

Consequently, $g(x) = x/\Phi_k(x) \to \infty$ as $x \downarrow 0$, so any minimizer of g is interior.

Proof. Because $k \geq 3$, $\psi_{k-1}(y) \leq \psi_2(y)$ for all $y \geq 0$. Moreover, for $y \geq 0$, $\psi_2(y) = 1 - e^{-y}(1+y) \leq \min\{1, y^2/2\}$. Hence

$$\Phi_k(x) = \mathbb{E}[W\psi_{k-1}(Wx)] \leq \mathbb{E}\big[W \min\{1, (Wx)^2/2\}\big] .$$

Splitting this upper bound on $W \leq 1/x$ and $W > 1/x$ gives

$$\Phi_k(x) \leq \frac{x^2}{2}\mathbb{E}[W^3 \mathbf{1}_{\{W \leq 1/x\}}] + \mathbb{E}[W \mathbf{1}_{\{W > 1/x\}}].$$

For the tail term, using $Wx > 1$ on $\{W > 1/x\}$, we have

$$\mathbb{E}[W \mathbf{1}_{\{W > 1/x\}}] \leq x\, \mathbb{E}[W^2 \mathbf{1}_{\{W > 1/x\}}] = o(x),$$

since $\mathbb{E}[W^2] < \infty$ implies $\mathbb{E}[W^2 \mathbf{1}_{\{W > t\}}] \to 0$ as $t \to \infty$.

Let $T = 1/x$ and consider the third truncated moment above. Using integration by parts,

$$\mathbb{E}[W^3 \mathbf{1}_{\{W \leq T\}}] = 3\int_0^T t^2\, \mathbb{P}(W > t)\, dt - T^3 \mathbb{P}(W > T) \;\leq\; 3\int_0^T t^2\, \mathbb{P}(W > t)\, dt.$$

Since $\mathbb{E}[W^2] < \infty$, we have $t^2 \mathbb{P}(W > t) \leq \mathbb{E}[W^2 \mathbf{1}_{\{W > t\}}] \to 0$. Therefore for any $\varepsilon > 0$ there exists T_0 such that $t^2 \mathbb{P}(W > t) \leq \varepsilon$ for all $t \geq T_0$, implying

$$\mathbb{E}[W^3 \mathbf{1}_{\{W \leq T\}}] \leq 3\int_0^{T_0} t^2 \mathbb{P}(W > t)\, dt + 3\varepsilon(T - T_0) = O(1) + o(T).$$

Thus $\mathbb{E}[W^3 \mathbf{1}_{\{W \leq T\}}] = o(T)$, so $(x^2/2)\mathbb{E}[W^3 \mathbf{1}_{\{W \leq 1/x\}}] = o(x)$. Combining terms gives $\Phi_k(x) = o(x)$, i.e. $\Phi_k(x)/x \to 0$, and hence $x/\Phi_k(x) \to \infty$. $\square$

OA.12.2 Derivative identities for g and the H-criterion

Assumption 2 is a regularity condition on g: it requires g to have a unique minimum and to be strictly increasing past it. These conditions ensure that the giant k-core emerges through a single discontinuous jump as connectivity increases. For some interaction profiles satisfying Assumption 1, however, g can have multiple local minima—in such cases, the giant k-core emerges through several disjoint jumps, one as connectivity crosses each local minimum of g. The lemma below characterizes the critical points of g via the auxiliary function $H(x) \equiv x\,\Phi_k'(x) - \Phi_k(x)$. By part (ii), g has a unique critical point on $(0,\infty)$ if and only if H crosses zero exactly once, which gives a tractable single-crossing diagnostic for when Assumption 2 holds.

Lemma 5 (H-criterion for extrema of g)**.** *Fix $k \geq 3$ and an interaction profile F satisfying Assumption 1. Let*

$$\Phi_k(x) \equiv \mathbb{E}[W\,\psi_{k-1}(Wx)]\,, \qquad g(x) \equiv \frac{x}{\Phi_k(x)} \quad (x > 0),$$

and define

$$H(x) \equiv x\,\Phi_k'(x) - \Phi_k(x).$$

Then:

(i) Φ_k is twice continuously differentiable on $(0,\infty)$, with derivatives

$$\Phi_k'(x) = \mathbb{E}\big[W^2\,p_{k-2}(Wx)\big]\,, \qquad \Phi_k''(x) = \mathbb{E}\big[W^3\,(p_{k-3}(Wx) - p_{k-2}(Wx))\big]\,,$$

where $p_m(z) \equiv \mathbb{P}(\text{Poisson}(z) = m)$.

(ii) g is continuously differentiable on $(0,\infty)$ and

$$g'(x) = \frac{\Phi_k(x) - x\Phi_k'(x)}{\Phi_k(x)^2} = -\frac{H(x)}{\Phi_k(x)^2}.$$

Consequently, $x > 0$ is a critical point of g if and only if $H(x) = 0$.

(iii) H is continuously differentiable on $(0,\infty)$ and satisfies the identity

$$H'(x) = x\,\Phi_k''(x).$$

Proof. Part (i): For each $w \geq 0$, we have

$$\frac{\partial}{\partial x}\Big[w\,\psi_{k-1}(wx)\Big] = w^2\,p_{k-2}(wx).$$

Because $p_{k-2}(z) \leq 1$ for all $z \geq 0$ and $\mathbb{E}[W^2] < \infty$, dominated convergence allows us to differentiate under the expectation, which gives

$$\Phi_k'(x) = \mathbb{E}\big[W^2\,p_{k-2}(Wx)\big]\,.$$

To calculate the second derivative, we first fix a compact interval $I = [a, b] \subset (0, \infty)$. For every integer $m \geq 0$ and every $z \geq 0$,

$$z\,p_m(z) = (m+1)p_{m+1}(z) \leq m+1.$$

Hence, for every $x \in I$,

$$W^3 p_m(Wx) = W^2\,\frac{Wx\,p_m(Wx)}{x} \leq \frac{m+1}{a}\,W^2.$$

Applying this inequality with $m = k-3$ and $m = k-2$ gives

$$W^3\big|p_{k-3}(Wx) - p_{k-2}(Wx)\big| \leq \frac{2k-3}{a}\,W^2,$$

which is integrable by Assumption 1. Since

$$\frac{\partial}{\partial x}\Big[W^2 p_{k-2}(Wx)\Big] = W^3\big(p_{k-3}(Wx) - p_{k-2}(Wx)\big),$$

dominated convergence gives

$$\Phi_k''(x) = \mathbb{E}\big[W^3\big(p_{k-3}(Wx) - p_{k-2}(Wx)\big)\big]\,, \qquad x \in I.$$

For each value of W, the integrand $W^3\,(p_{k-3}(Wx) - p_{k-2}(Wx))$ is continuous in x. For every $x \in I$, its absolute value is bounded by the same random variable $\frac{2k-3}{a}\,W^2$, which has finite expectation. Thus, as x approaches any point of I, dominated convergence allows us to pass the limit through the expectation. This shows that Φ_k'' is continuous on I. Since $I \subset (0, \infty)$ was arbitrary, $\Phi_k \in C^2((0, \infty))$.

Part (ii): This is the quotient rule:

$$g'(x) = \frac{\Phi_k(x) - x\Phi_k'(x)}{\Phi_k(x)^2} = -\frac{H(x)}{\Phi_k(x)^2}.$$

Hence $g'(x) = 0$ if and only if $H(x) = 0$.

Part (iii): By definition,

$$H(x) = x\Phi_k'(x) - \Phi_k(x),$$

so, using Part (i),

$$H'(x) = \Phi_k'(x) + x\Phi_k''(x) - \Phi_k'(x) = x\Phi_k''(x).$$

□

OA.12.3 Uniform small-x bound under bounded dispersion

Lemma 6 (Uniform small-x bound under bounded dispersion)**.** *Fix $k \geq 3$ and let $W \sim F$ satisfy $\mathbb{E}[W] = 1$ and $\mathrm{Var}(W) = \sigma^2 < \infty$. Then for all $x \in (0, 1/2]$,*

$$\Phi_k(x) = \mathbb{E}[W\psi_{k-1}(Wx)] \ \leq \ 4x^2 + 6\sigma^2 x. \tag{30}$$

Consequently, for all $x \in (0, 1/2]$,

$$g(x) = \frac{x}{\Phi_k(x)} \ \geq \ \frac{1}{4x + 6\sigma^2}. \tag{31}$$

In particular, for any $a \in (0, 1/2]$,

$$\inf_{0<x\leq a} g(x) \ \geq \ \frac{1}{4a + 6\sigma^2}.$$

Proof. Because $k \geq 3$ we have $\psi_{k-1}(y) \leq \psi_2(y)$ for all $y \geq 0$. Moreover, if $N \sim \mathrm{Poisson}(y)$ then $1_{\{N\geq 2\}} \leq N(N-1)/2$, so

$$\psi_2(y) = \mathbb{P}(N \geq 2) \leq \frac{\mathbb{E}[N(N-1)]}{2} = \frac{y^2}{2},$$

and hence $\psi_2(y) \leq \min\{1, y^2/2\}$. Therefore,

$$\Phi_k(x) \leq \mathbb{E}\Big[W \min\Big\{1, \frac{(Wx)^2}{2}\Big\}\Big] \leq \frac{x^2}{2}\mathbb{E}\big[W^3 \mathbf{1}_{\{W \leq 1/x\}}\big] + \mathbb{E}\big[W \mathbf{1}_{\{W > 1/x\}}\big] .$$

Assume $x \leq 1/2$, so $1/x \geq 2$. For the first term, split at $W \leq 2$:

$$\mathbb{E}\big[W^3 \mathbf{1}_{\{W \leq 1/x\}}\big] \leq \mathbb{E}\big[W^3 \mathbf{1}_{\{W \leq 2\}}\big] + \mathbb{E}\big[W^3 \mathbf{1}_{\{2 < W \leq 1/x\}}\big] \leq 8 + \frac{1}{x}\mathbb{E}\big[W^2 \mathbf{1}_{\{W > 2\}}\big] ,$$

since $W^3 \leq (1/x)W^2$ on $\{W \leq 1/x\}$. On $\{W > 2\}$ we have $W - 1 \geq 1$, so Chebyshev implies $\mathbb{P}(W > 2) \leq \sigma^2$ and hence

$$\mathbb{E}\big[W^2 \mathbf{1}_{\{W > 2\}}\big] \leq 2\mathbb{E}[(W-1)^2] + 2\mathbb{P}(W > 2) \leq 4\sigma^2.$$

Thus

$$\frac{x^2}{2}\mathbb{E}\big[W^3 \mathbf{1}_{\{W \leq 1/x\}}\big] \leq 4x^2 + 2\sigma^2 x.$$

For the second term, write $W = 1 + (W - 1)$ and use Cauchy–Schwarz and Chebyshev: for $t > 1$,

$$\mathbb{E}[W \mathbf{1}_{\{W > t\}}] \leq \mathbb{E}[(W-1)\mathbf{1}_{\{W > t\}}] + \mathbb{P}(W > t) \leq \sigma\sqrt{\mathbb{P}(W > t)} + \mathbb{P}(W > t) \leq \frac{\sigma^2}{t-1} + \frac{\sigma^2}{(t-1)^2}.$$

Taking $t = 1/x$ yields

$$\mathbb{E}\big[W \mathbf{1}_{\{W > 1/x\}}\big] \leq \frac{\sigma^2 x}{1-x} + \frac{\sigma^2 x^2}{(1-x)^2} \leq 4\sigma^2 x,$$

where the last inequality uses $x \leq 1/2$. Combining bounds gives (30). Dividing by x yields (31). □

OA.12.4 Uniform approximation on compacts

Lemma 7 (Uniform approximation on compacts)**.** *Fix $k \geq 3$ and $X > 0$, and let $W \sim F$ satisfy $\mathbb{E}[W] = 1$. Then*

$$\sup_{0 \leq x \leq X} \big|\Phi_k(x) - \psi_{k-1}(x)\big| \leq (1 + X)\,\mathbb{E}|W - 1|, \tag{32}$$

$$\sup_{0 \leq x \leq X} \big|A_k(x) - \psi_k(x)\big| \leq X\,\mathbb{E}|W - 1|. \tag{33}$$

If additionally $\mathbb{E}[W^2] < \infty$*, then*

$$\sup_{0\le x\le X} \left|\Phi_k'(x) - \psi_{k-1}'(x)\right| \le \mathbb{E}|W^2 - 1| + X\,\mathbb{E}|W-1|. \tag{34}$$

Proof. We use only that ψ_m and p_m are 1-Lipschitz in their argument: since $\psi_m'(y) = p_{m-1}(y) \in [0,1]$ for $y \ge 0$, we have $|\psi_m(y) - \psi_m(z)| \le |y-z|$; similarly $p_m'(y) = p_{m-1}(y) - p_m(y) \in [-1,1]$ so $|p_m(y) - p_m(z)| \le |y-z|$.

Bound (32)*.* Write

$$\Phi_k(x) - \psi_{k-1}(x) = \mathbb{E}\big[W\psi_{k-1}(Wx) - \psi_{k-1}(x)\big].$$

Add and subtract $\psi_{k-1}(Wx)$:

$$W\psi_{k-1}(Wx) - \psi_{k-1}(x) = (W-1)\psi_{k-1}(Wx) + \big(\psi_{k-1}(Wx) - \psi_{k-1}(x)\big).$$

Since $\psi_{k-1}(\cdot) \in [0,1]$, the first term has absolute value at most $|W-1|$. Because ψ_{k-1} is Lipschitz, the second term is bounded by $|Wx - x| = x|W-1|$. Taking expectations yields $|\Phi_k(x) - \psi_{k-1}(x)| \le (1+x)\mathbb{E}|W-1|$, and taking $x \le X$ gives (32).

Bound (33)*.* Because ψ_k is Lipschitz,

$$|A_k(x) - \psi_k(x)| = \big|\mathbb{E}[\psi_k(Wx) - \psi_k(x)]\big| \le \mathbb{E}[|Wx - x|] = x\,\mathbb{E}|W-1| \le X\,\mathbb{E}|W-1|.$$

Bound (34)*.* Differentiate Φ_k:

$$\Phi_k'(x) = \mathbb{E}\big[W^2\,\psi_{k-1}'(Wx)\big] = \mathbb{E}\big[W^2\,p_{k-2}(Wx)\big], \qquad \psi_{k-1}'(x) = p_{k-2}(x).$$

Then

$$\Phi_k'(x) - p_{k-2}(x) = \mathbb{E}\big[(W^2-1)p_{k-2}(Wx)\big] + \mathbb{E}\big[p_{k-2}(Wx) - p_{k-2}(x)\big].$$

The first term is bounded by $\mathbb{E}|W^2 - 1|$ since $p_{k-2} \in [0,1]$. The second term is bounded by $\mathbb{E}|Wx - x| = x\mathbb{E}|W-1| \le X\mathbb{E}|W-1|$ because p_{k-2} is Lipschitz. This yields (34). □

OA.12.5 Assumption 2 for sufficiently homogeneous F

Lemma 8. *Fix $k \geq 3$. There exists $\varepsilon_3 > 0$ such that any interaction profile F satisfying Assumption 1 and $\sigma(F) \leq \varepsilon_3$ also satisfies Assumption 2, and the unique minimizer $x_k^\star(F)$ of $g^F(x) \equiv x/\Phi_k(x)$ satisfies $x_k^\star(F) \to x_k^{\star,ER}$ as $\sigma(F) \to 0$.*

Proof. Write $g^F(x) \equiv x/\Phi_k(x)$ and $g^{ER}(x) \equiv x/\psi_{k-1}(x)$, and let $x^\star \equiv x_k^{\star,ER}$ denote the unique minimizer of g^{ER} (existence and uniqueness are established in Step (a) below). Define

$$H^F(x) \equiv x\,\Phi_k'(x) - \Phi_k(x), \qquad H^{ER}(x) \equiv x\,\psi_{k-1}'(x) - \psi_{k-1}(x),$$

so that by Lemma 5(ii),

$$(g^F)'(x) = -\frac{H^F(x)}{\Phi_k(x)^2}, \qquad (g^{ER})'(x) = -\frac{H^{ER}(x)}{\psi_{k-1}(x)^2}.$$

The proof has four steps: (a) the strict sign pattern of H^{ER} on $(0,\infty)$, by direct calculation; (b) transfer of this sign pattern to H^F on a compact interval $[a,b]$, giving at least one zero of H^F near $x^\star$; (c) strict monotonicity of H^F on $[x^\star-\delta, b]$, upgrading existence to uniqueness; and (d) extension of strict increase of g^F from $[x_k^\star(F), b]$ to all of $[x_k^\star(F), \infty)$.

Step (a): sign pattern of H^{ER} (by direct calculation). Since $\psi_{k-1}'(x) = p_{k-2}(x)$, we have $H^{ER}(x) = x\,p_{k-2}(x) - \psi_{k-1}(x)$. Differentiating and using $p_m'(x) = p_{m-1}(x) - p_m(x)$ gives $(H^{ER})'(x) = p_{k-2}(x) + x\,p_{k-2}'(x) - \psi_{k-1}'(x) = x(p_{k-3}(x) - p_{k-2}(x))$. Moreover $p_{k-3}(x)/p_{k-2}(x) = (k-2)/x$, so $p_{k-3}(x) - p_{k-2}(x)$ is strictly positive for $x < k-2$ and strictly negative for $x > k-2$. Hence H^{ER} is strictly increasing on $(0, k-2]$ and strictly decreasing on $[k-2, \infty)$. Since $H^{ER}(x) \to 0$ as $x \downarrow 0$ and H^{ER} is strictly increasing on $(0, k-2]$, we have $H^{ER}(k-2) > 0$; and $H^{ER}(x) \to -1$ as $x \to \infty$ because $p_{k-2}(x) \to 0$ and $\psi_{k-1}(x) \to 1$. Therefore H^{ER} has a unique zero on $(0,\infty)$, necessarily at some $x^\star > k-2$. In particular, $H^{ER}(x) > 0$ for $x \in (0, x^\star)$ and $H^{ER}(x) < 0$ for $x > x^\star$, so g^{ER} is strictly decreasing on $(0, x^\star)$ and strictly increasing on $(x^\star, \infty)$.

Step (b): transfer the single-crossing of H^{ER} to H^F on a compact interval. By Step (a), $x^\star > k-2$. Choose $\delta > 0$ with $x^\star - \delta > k-2$, a with $0 < a < x^\star - \delta$ (e.g., $a = (k-2)/2$), and b with $b > x^\star + \delta$ large enough that $\psi_{k-1}(b/2) \geq 1/2$ and $k^k e^{-k}/((k-2)!\,b) \leq 1/8$ (we use these bounds in Step (d)). By continuity and the strict sign pattern in Step (a),

$$m_L \equiv \min_{x\in[a,x^\star-\delta]} H^{ER}(x) > 0, \qquad m_R \equiv \min_{x\in[x^\star+\delta,b]} \big(- H^{ER}(x)\big) > 0.$$

Next note the deterministic bound, for all $x \in [a, b]$,

$$|H^F(x)-H^{ER}(x)| \le x\,|\Phi_k'(x)-\psi_{k-1}'(x)|+|\Phi_k(x)-\psi_{k-1}(x)| \le b \sup_{[0,b]} |\Phi_k'-\psi_{k-1}'|+\sup_{[0,b]} |\Phi_k-\psi_{k-1}|.$$

By Lemma 7 (and $\mathbb{E}|W-1| \le \sigma(F)$, $\mathbb{E}|W^2-1| \le \sigma(F)\sqrt{4+\sigma(F)^2}$), the right-hand side tends to 0 as $\sigma(F) \to 0$. Hence, for $\sigma(F)$ sufficiently small,

$$H^F(x) \ge \frac{m_L}{2} \text{ for all } x \in [a, x^\star-\delta], \qquad H^F(x) \le -\frac{m_R}{2} \text{ for all } x \in [x^\star+\delta, b]. \tag{35}$$

In particular $H^F(x^\star-\delta) > 0$ and $H^F(x^\star+\delta) < 0$, so by continuity H^F has at least one zero in $(x^\star-\delta, x^\star+\delta)$.

Step (c): uniqueness via monotonicity of H^F on $[x^\star-\delta, b]$. For $x > 0$, differentiating $\Phi_k'(x) = \mathbb{E}[W^2 p_{k-2}(Wx)]$ under the expectation (as in Lemma 5(i)) gives $\Phi_k''(x) = \mathbb{E}[W^3(p_{k-3}(Wx) - p_{k-2}(Wx))]$. Let $q(z) \equiv p_{k-3}(z) - p_{k-2}(z) = \psi_{k-1}''(z)$. Because $x^\star - \delta > k-2$, q is continuous and strictly negative on $[x^\star-\delta, b]$, so $m'' \equiv \min_{x\in[x^\star-\delta,b]}(-q(x)) > 0$. For each $x \in [x^\star-\delta, b]$, define $f_x(w) \equiv w^3 q(wx)$. Using the explicit Poisson form $p_m(t) = e^{-t}t^m/m!$, every term in $f_x'(w)$ is a polynomial in w times e^{-wx}; since $x \ge x^\star-\delta > 0$, this implies

$$L \;\equiv\; \sup_{x\in[x^\star-\delta,b]} \;\sup_{w\ge 0}\; |f_x'(w)| \;<\; \infty,$$

because $\sup_{w\ge0} w^m e^{-w(x^\star-\delta)} < \infty$ for each fixed m. Therefore, for all $x \in [x^\star-\delta, b]$,

$$|\Phi_k''(x) - q(x)| = |\mathbb{E}[f_x(W) - f_x(1)]| \le L\,\mathbb{E}|W-1| \le L\,\sigma(F).$$

Choosing $\sigma(F)$ small enough that $L\sigma(F) \le m''/2$ yields $\Phi_k''(x) \le -m''/2 < 0$ on $[x^\star-\delta, b]$. Consequently, by Lemma 5(iii), $(H^F)'(x) = x\,\Phi_k''(x) < 0$ for all $x \in [x^\star-\delta, b]$, so H^F is strictly decreasing there. Combined with (35), this implies that H^F has a *unique* zero in $(x^\star-\delta, x^\star+\delta)$; denote it by $x_k^\star(F)$. Since $(g^F)'(x) = -H^F(x)/\Phi_k(x)^2$, we have $(g^F)'(x) < 0$ for $x < x_k^\star(F)$ and $(g^F)'(x) > 0$ for $x \in (x_k^\star(F), b]$, hence g^F is strictly decreasing on $[a, x_k^\star(F)]$ and strictly increasing on $[x_k^\star(F), b]$. Moreover, since

$\delta > 0$ was arbitrary, this shows $x_k^\star(F) \to x^\star$ as $\sigma(F) \to 0$.

Step (d): strict increase on $[x_k^\star(F), \infty)$. Step (c) gives strict increase on $[x_k^\star(F), b]$. We extend this to $[b, \infty)$ using the conditions $\psi_{k-1}(b/2) \geq 1/2$ and $k^k e^{-k}/((k-2)!\, b) \leq 1/8$ chosen in Step (b). Then for any F satisfying Assumption 1 and any $x \geq b$,

$$\Phi_k(x) \geq \mathbb{E}\big[W\, \psi_{k-1}(Wx)\mathbf{1}_{\{W \geq 1/2\}}\big] \geq \psi_{k-1}(b/2)\, \mathbb{E}\big[W\mathbf{1}_{\{W \geq 1/2\}}\big] \geq \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4},$$

where we used $Wx \geq x/2 \geq b/2$ on $\{W \geq 1/2\}$ and $\mathbb{E}[W\mathbf{1}_{\{W<1/2\}}] \leq 1/2$. Also, using $p_{k-2}(t) = e^{-t}t^{k-2}/(k-2)!$ and $\sup_{w \geq 0} w^k e^{-wx} = (k/x)^k e^{-k}$, let $C_k \equiv k^k e^{-k}/(k-2)!$. Then

$$x\Phi_k'(x) = x\, \mathbb{E}[W^2 p_{k-2}(Wx)] = \frac{x^{k-1}}{(k-2)!}\, \mathbb{E}\big[W^k e^{-Wx}\big] \leq \frac{x^{k-1}}{(k-2)!} \cdot \frac{k^k e^{-k}}{x^k} = \frac{C_k}{x} \leq \frac{C_k}{b} \leq \frac{1}{8}.$$

Thus $\Phi_k(x) - x\Phi_k'(x) \geq 1/8 > 0$ for all $x \geq b$, so $(g^F)'(x) > 0$ on $[b, \infty)$. Therefore g^F is strictly increasing on $[x_k^\star(F), \infty)$.

By Steps (a)–(d), g^F has the unique minimizer $x_k^\star(F)$ and is strictly increasing on $[x_k^\star(F), \infty)$, so Assumption 2 holds. The convergence $x_k^\star(F) \to x^\star$ was established in Step (c).

$\square$

OA.12.6 Existence of the threshold $\bar{\beta}$

Lemma 9. *Suppose $\alpha > c_k(F)$. Define $f(\beta) \equiv \sup_{r > c_k(F)} \pi_\beta(r; F)$. There exists a unique $\bar{\beta}(\alpha; F) \in (0, \infty)$ such that*

$$f(\bar{\beta}) = u(c_k(F)), \qquad f(\beta) > u(c_k(F)) \text{ for } \beta < \bar{\beta}, \qquad f(\beta) < u(c_k(F)) \text{ for } \beta > \bar{\beta},$$

where $u(r) \equiv \alpha r - \frac{1}{2}r^2$.

Proof. Write $c \equiv c_k(F)$ and $J \equiv J_k(F) > 0$. By Theorem 2(i) and the definition of J, $a(r) \geq J$ for all $r > c$. Hence for any $0 \leq \beta_1 < \beta_2$ and any $r > c$, $\pi_{\beta_1}(r; F) - \pi_{\beta_2}(r; F) = (\beta_2 - \beta_1)a(r) \geq (\beta_2 - \beta_1)J$. Taking suprema gives $f(\beta_1) \geq f(\beta_2) + (\beta_2 - \beta_1)J > f(\beta_2)$, so f is strictly decreasing. It is also finite and convex (as the pointwise supremum of affine functions of β), and hence continuous on $(0, \infty)$.

Recall $r^{\text{naive}} = \alpha > c$ is the unconstrained maximizer of u. Then $f(0) = \sup_{r > c} u(r) = u(r^{\text{naive}}) > u(c)$, while $f(\beta) \leq u(r^{\text{naive}}) - \beta J \to -\infty$ as $\beta \to \infty$.

By the intermediate value theorem and strict monotonicity, there exists a unique $\bar{\beta} \in (0, \infty)$ with the stated properties. $\square$

OA.12.7 Smoothness and strict monotonicity of participation above the cutoff

Lemma 10 (Smoothness of participation above the cutoff)**.** *Under Assumption 2 the following holds: For each $r > c \equiv c_k(F)$, let $x(r)$ be the unique solution to $x = r\Phi_k(x)$ in $[x_k^\star(F), \infty)$, and recall $a(r) \equiv a_k(r; F) = A_k(x(r))$.*

Then $r \mapsto x(r)$ and $r \mapsto a(r)$ are C^2 on (c, ∞). Moreover, for all $r > c$,

$$a'(r) = A_k'(x(r))\, x'(r) > 0.$$

In particular, $a(\cdot)$ is strictly increasing on (c, ∞), and for each $\beta > 0$ the map $r \mapsto \pi_\beta(r; F) = \alpha r - \frac{1}{2}r^2 - \beta\, a(r)$ is C^2 on (c, ∞).

Proof. Write $p_m(z) \equiv \mathbb{P}(\text{Poisson}(z) = m) = e^{-z}z^m/m!$.

Φ_k is C^2 on $(0, \infty)$. By Lemma 5(i), $\Phi_k \in C^2((0, \infty))$ with derivatives $\Phi_k'(x) = \mathbb{E}[W^2\, p_{k-2}(Wx)]$ and $\Phi_k''(x) = \mathbb{E}[W^3(p_{k-3}(Wx) - p_{k-2}(Wx))]$.

A_k is C^2 on $(0, \infty)$. Since $\psi_k'(z) = p_{k-1}(z)$ and $|p_{k-1}(z)| \leq 1$, dominated convergence (with integrable dominator W) gives, for every $x > 0$, $A_k'(x) = \mathbb{E}[W\, p_{k-1}(Wx)]$. For the second derivative, $\partial_x[W\, p_{k-1}(Wx)] = W^2(p_{k-2}(Wx) - p_{k-1}(Wx))$, and $W^2|p_{k-2}(Wx) - p_{k-1}(Wx)| \leq 2W^2$, which is integrable by $\mathbb{E}[W^2] < \infty$ (Assumption 1). Dominated convergence therefore yields $A_k''(x) = \mathbb{E}[W^2(p_{k-2}(Wx) - p_{k-1}(Wx))]$, and A_k'' is continuous by the same domination, so $A_k \in C^2((0, \infty))$.

For $r > c$, Assumption 2 implies that the equation $r = g(x)$ has a unique solution $x = x(r) \in [x_k^\star(F), \infty)$, and this $x(r)$ is exactly the largest-fixed-point branch solving $x = r\Phi_k(x)$. Since $c = g(x_k^\star(F))$ and g is strictly increasing on $[x_k^\star(F), \infty)$, we have $x(r) > x_k^\star(F)$ for all $r > c$. By Assumption 2, $g'(x(r)) > 0$. Now $g'(x) = (\Phi_k(x) - x\Phi_k'(x))/\Phi_k(x)^2$, so $g'(x(r)) > 0$ implies $\Phi_k(x(r)) > x(r)\Phi_k'(x(r))$. Using $x(r) = r\Phi_k(x(r))$, this is equivalent to $1 - r\Phi_k'(x(r)) > 0$. Hence the implicit function theorem applied to $H(r, x) \equiv x - r\Phi_k(x) = 0$ gives that $r \mapsto x(r)$ is C^2 on (c, ∞), with derivative $x'(r) = \Phi_k(x(r))/(1 - r\Phi_k'(x(r))) > 0$. Since A_k is C^2, $a(r) = A_k(x(r))$ is C^2 on (c, ∞), and $a'(r) = A_k'(x(r))\, x'(r)$. Finally, $A_k'(x(r)) > 0$ because $\mathbb{P}(W > 0) > 0$ (Assumption 1) and $p_{k-1}(Wx(r)) > 0$ whenever $W > 0$ and $x(r) > 0$. Thus $a'(r) > 0$ for all $r > c$. $\square$

OA.12.8 Generic uniqueness of the principal's optimum

Lemma 11 (Generic uniqueness)**.** *Fix $\alpha > 0$ and define the value function*

$$V(\beta) \equiv \sup_{r \geq 0} \Big(\alpha r - \tfrac{1}{2} r^2 - \beta\, a_k(r; F)\Big).$$

Then V is convex in β, hence differentiable for all but countably many β. Moreover, if V is differentiable at β, then $\arg\max_{r \geq 0} \pi_\beta(r; F)$ *is a singleton.*

Proof. For each fixed r, $\beta \mapsto \pi_\beta(r; F)$ is affine with slope $-a_k(r; F)$, so $V(\beta) = \sup_r \pi_\beta(r; F)$ is convex as the pointwise supremum of affine functions. A convex function on an interval is differentiable except at a countable set of points.

Now fix β and suppose there are two distinct global maximizers $r_1 < r_2$. If $r_2 \leq c_k(F)$, then $a_k(r_1; F) = a_k(r_2; F) = 0$ and $\pi_\beta(r; F) = \alpha r - \frac{1}{2} r^2$ on $[0, c_k(F)]$, which has a unique maximizer on $[0, c_k(F)]$ (equal to $\min\{\alpha, c_k(F)\}$), a contradiction. Hence $r_2 > c_k(F)$, and by strict monotonicity of $a_k(\cdot; F)$ on $(c_k(F), \infty)$ (Theorem 2(i)), we have $a_k(r_1; F) \neq a_k(r_2; F)$. Therefore the two affine functions $\beta \mapsto \pi_\beta(r_1; F)$ and $\beta \mapsto \pi_\beta(r_2; F)$ have different slopes, so V cannot be differentiable at β. $\square$